Square-wave excitation of a linear oscillator

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Abstract. The paper deals with forced oscillations of a torsion spring pendulum excited by an external square-wave driving torque. Two different ways of determining the steady-state response of the oscillator to a non-harmonic driving force are described and compared. Behavior of this familiar mechanical system can help a student to better understand why and how an electromagnetic oscillatory LCR-circuit transfers the square-wave voltage from input to output with a distortion of its shape.

1. Introduction: the physical system

Most textbooks on general physics treat forced oscillations in a linear system under a sinusoidal driving force rather extensively (see, for example, Berkeley Physics Course [1], [2], [3]). The general case of a periodic but non-sinusoidal excitation of a linear oscillator is usually only mentioned with a reference to the principle of superposition and an expansion of an arbitrary periodic force as the Fourier series of sine and cosine functions. In this paper an alternative approach to the problem of forced oscillations is suggested and compared with the traditional treatment.

To study forced oscillations caused in a linear system by a non-sinusoidal periodic external influence, we employ a simplified model of a torsion spring oscillator. Its schematic image is shown on the left side of Fig. 1. The oscillator is similar to the balance devices of ordinary mechanical watches—a balanced massive rotor *1* (flywheel) attached to one end of an elastic spiral spring *2*. The spring provides a restoring torque proportional to the angular displacement of the flywheel from the equilibrium position. To provide an external excitation, the other end of the spiral spring is attached to a driving rod *3* that can be turned about the axis common with the axis of the flywheel. When the rod is constrained to move periodically to and fro about some middle position, an additional periodic torque is exerted on the flywheel. This mode of excitation is called *kinematical* because it is characterized by a given motion of some part of the system rather than by a given external torque.

An external square-wave torque can be realized by abruptly displacing the driving rod alternately in opposite directions through the same angle in equal time intervals. We suppose that the displacements of the rod and thus of the equilibrium position of the flywheel occur so quickly that there is no significant change in either the angular position or velocity of the flywheel during the displacement of the rod.

The right-hand side of Fig. 1 shows a LCR-oscillatory circuit that can be regarded as an electromagnetic analogue of the mechanical device. Both systems are described by identical differential equations and thus are dynamically isomorphic. However,
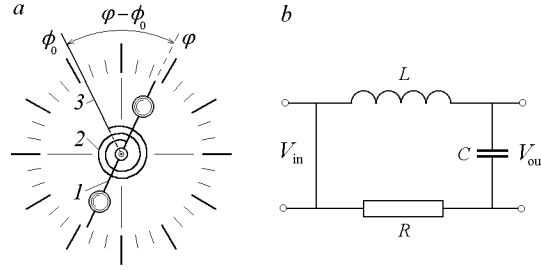
Figure 1. Schematic image of the torsion spring oscillator (left) and its electromagnetic analogue—LCR-oscillatory circuit excited by the square-wave input voltage (right).

the mechanical system has a definite didactic advantage for exploration of forced oscillations because it allows us to observe a direct visualization of motion.

This system is simulated in one of the programs of the educational software package PHYSICS OF OSCILLATIONS developed by the author [4]. All the graphs that illustrate the paper are obtained with the help of this program.

2. The differential equation of forced oscillations

When the system is at rest, the rod of the flywheel is parallel to the driving rod and the spring is not strained. The zero point of the dial indicates the central position of the exciting rod (the vertical position in Fig. 1). The angle of deflection φ indicates the momentary position of the flywheel. When the rod is deflected from the vertical position through an angle ϕ, the spiral spring is twisted from its unstrained state through the angle $\varphi - \phi$. The spring then exerts a torque $-D(\varphi - \phi)$ on the flywheel, where D is the torsion spring constant. Thus, the differential equation of rotation of the flywheel, whose moment of inertia about the axis of rotation is I, is given by

$$I\ddot{\varphi} = -D(\varphi - \phi). \tag{1}$$

We transfer $-D\varphi$ (the part of the elastic torque which is proportional to φ) to the left side of Eq. (1), divide the resulting equation by I, and introduce the value $\omega_0 = \sqrt{D/I}$, whose physical meaning is the frequency of natural oscillations in the absence of friction. Thus we obtain:

$$\ddot{\varphi} + \omega_0^2 \varphi = \omega_0^2 \phi. \tag{2}$$

The right-hand side $\omega_0^2 \phi$ of this equation can be treated as an external torque (divided by I) caused by the displacement of the driving rod from its central position through an angle ϕ. We let the instantaneous displacements of the rod occur alternately to the right and to the left after the lapse of equal time intervals $T/2$, so that T is the full period of the external non-sinusoidal action, repeated indefinitely. In the presence of viscous friction, a term $2\gamma\dot{\varphi}$ proportional to the angular velocity $\dot{\varphi}$ should be added to Eq. (2), in which the damping constant γ characterizes the strength of viscous friction in the system:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \varphi = \begin{cases} \omega_0^2 \phi_0, & (0,\ T/2), \\ -\omega_0^2 \phi_0, & (T/2,\ T). \end{cases} \tag{3}$$

Forced oscillations of the electric charge q stored in a capacitor of a resonant series LCR-circuit (see the right-hand side of Fig. 1) excited by a square-wave input voltage $V_{\text{in}}(t)$ obey the same differential equation as does the forced oscillation of a mechanical torsion spring oscillator excited by periodic abrupt changes of position of the driving rod:

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = \omega_0^2 C V_{\text{in}}(t). \tag{4}$$

In this equation ω_0 is the natural frequency of oscillations of charge in the circuit in the absence of resistance. It depends on the capacitance C of the capacitor and the inductance L of the coil: $\omega_0 = 1/\sqrt{LC}$. The damping constant $\gamma = R/(2L)$ characterizes the dissipation of electromagnetic energy occurring in a resistor whose resistance is R (see, for example, Ref. [3], Chapter 8).

Because of this similarity, the mechanical system described above enables us to give a very clear explanation for transformation of the square-wave input voltage $V_{\text{in}}(t) = \pm V_0$ into the output voltage $V_{\text{out}}(t) = V_C(t) = q/C$ (voltage across the capacitor C), whose time dependence differs considerably from the piecewise constant input voltage. The output voltage $V_C(t)$ is analogous to the angular displacement $\varphi(t)$ of the rotor, while the alternating electric current $I(t) = \dot{q}(t)$ in the circuit is analogous to the angular velocity $\dot{\varphi}(t)$. However, some caution is necessary in interpreting the analogy between the mechanical oscillator and the electric LCR-circuit with respect to the energy transformations [5].

3. Harmonics of the driving force and of the steady-state response

Because of friction, natural oscillations of the flywheel gradually damp out, and a steady-state periodic motion of the flywheel is eventually established with a period equal to the period T of the driving force. The greater the decay time, $\tau = 1/\gamma$, of natural oscillations, the longer the duration of this transient process.

In the case of a sinusoidal driving torque, the steady-state oscillations of the flywheel acquire not only the period of the external action but also the same sinusoidal time dependence. However, a periodic driving force whose time dependence is something other than a pure sinusoid, produces a steady-state response which has the same period but whose time dependence differs from that of the driving force.

We consider below two different ways of determining the steady-state response of the oscillator to a non-harmonic square-wave driving force. One (traditional) way is based on decomposition of the external force time dependence in a Fourier series, i.e., on the representation of this force as a superposition of sinusoidal components (harmonics). Because the differential equation of motion for the spring oscillator is linear, each sinusoidal component of the driving torque (the *input harmonic*) produces its own sinusoidal response of the same frequency in the motion of the flywheel (the *output harmonic*), whose amplitude and phase can be calculated separately. The corresponding formulas are the same as for the familiar case of monoharmonic excitation (see, for example, [1], [3], or [6]). The net steady-state forced motion of the flywheel can be found as a superposition of these individual responses. Since the relative contributions of harmonic components to this response differ from their contributions to the driving force, the graph of motion of the flywheel has a different shape than that of the driving rod.

In particular, it may occur that one of the input harmonics with relatively small amplitude induces an especially large amplitude in the output oscillations. Such is

the case when the frequency of this harmonic is close to the natural frequency ω_0 of the oscillator since forced oscillations caused by this sinusoidal force occur under conditions of resonance. On the other hand, the relative contributions of the input harmonics whose frequencies lie far from the maximum of the resonance curve, are considerably attenuated in the output oscillations. The oscillator responds *selectively* to sinusoidal external forces of different frequencies. The phenomenon of resonance occurs only if the input spectrum contains a harmonic component whose frequency is close to the natural frequency of the oscillator.

Differences between the time dependence of output steady oscillations and that of the input driving force (distortions of the signal from input to output) are caused not only by changes in the relative amplitudes of different harmonics but also by changes in their phases from input to output. In the case of weak damping the dependence of phase on frequency is nearly a step-function. Specifically, all harmonic components whose frequencies $\omega_k = k\omega = (2\pi/T)k$ (T – driving period) are lower than the natural frequency ω_0, contribute to the output oscillations of the flywheel nearly in the same phases as they do to the input driving force. But harmonics whose frequencies ω_k are higher than the natural frequency contribute to the output oscillations with nearly inverted phases. The sinusoidal component whose frequency equals ω_0 lags in phase by $\pi/2$ behind the corresponding harmonic in the spectrum of the driving force.

The analytic expression for Eq. (3) in which the square-wave right-hand side has been Fourier decomposed has the following form:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \sum_{k=1,\,3,\,5\dots}^{\infty} \frac{4\phi_0\omega_0^2}{\pi k}\,\sin\omega_k t. \tag{5}$$

The Fourier series of the square-wave external force in Eq. (3) contains only odd-number harmonics with frequencies $\omega_k = k\omega$ ($k = 1,\,3,\,5,\,\dots$), where $\omega = 2\pi/T$ is the frequency of the driving force. We note that the amplitudes of harmonics of the square-wave function decrease rather slowly, as $1/k$, with the increase of their index k and their frequency ω_k: The frequency spectrum of the square-wave driving force is rich in harmonics.

For each sinusoidal term in the right-hand side of Eq. (5) the periodic particular solution is given by commonly known expressions (see, for example, [6]). Adding these solutions, we get the following time dependence of the angular displacement, $\varphi(t)$, for steady-state forced oscillations under the square-wave excitation:

$$\varphi(t) = \sum_{k=1,\,3,\,5\dots}^{\infty} \frac{4\phi_0}{\pi k} \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_k^2)^2 + 4\gamma^2\omega_k^2}}\,\sin(\omega_k t + \alpha_k), \tag{6}$$

where the phases α_k of the individual harmonics are determined by

$$\tan\alpha_k = \frac{2\gamma\omega_k}{\omega_k^2 - \omega_0^2}. \tag{7}$$

Equations (6) and (7) display clearly the above discussed peculiarities of the oscillator response to the square-wave driving action of the rod. A resonant response from the oscillator occurs each time the denominator in one of the terms of the sum in Eq. (6) is minimal, that is, when the frequency ω_k of one of the harmonics of the external force is equal to the resonant frequency ω_{res} of the oscillator:

$$\omega_{\mathrm{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0\left(1 - \frac{\gamma^2}{\omega_0^2}\right).$$

The latter approximate expression for ω_{res} is valid for a weakly damped oscillator ($\gamma \ll \omega_0$), whose quality factor Q is large ($Q = \omega_0/2\gamma \gg 1$). Since the fractional difference between ω_{res} and ω_0 is of the second order in the small parameter $\gamma/\omega_0 = 1/(2Q)$, in most cases of practical importance we need not distinguish the resonant frequency from the natural one and can assume that $\omega_{\text{res}} = \omega_0$. For $\omega_k < \omega_0$ Eq. (7) yields $\alpha_k \approx 0$, which means that the corresponding harmonic contributes to the output oscillations in the same phase as to the input square-wave force. On the contrary, for $\omega_k > \omega_0$ Eq. (7) yields $\alpha_k \approx -\pi$, and this harmonic component enters into the output oscillations with the inverted phase.
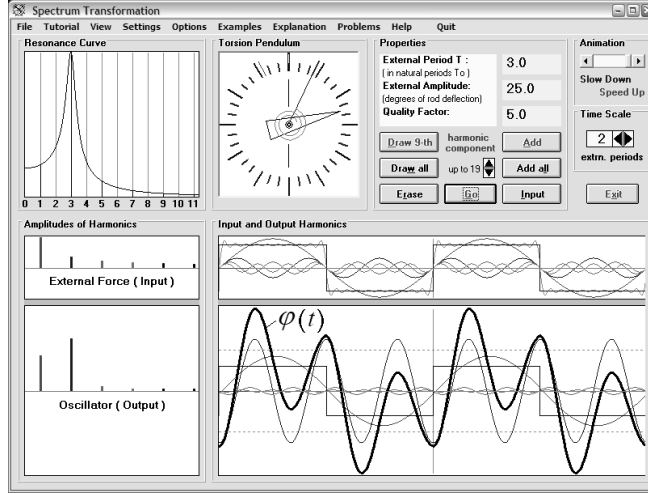

Figure 2. Transformation of the spectrum of the input square-wave external torque into the spectrum of steady-state output oscillations (see text for detail).

Figure 2 illustrates the transformation of the input spectrum of an external square-wave force into the output spectrum of the steady-state response of the oscillator for the case in which the driving period equals three natural periods. Since the third harmonic occurs under the maximum of the resonance curve (see the left upper corner in Fig. 2), this harmonic dominates the output spectrum. The time-dependent graphs of the input and output harmonics and of their sums are shown in the right lower part of Fig. 2.

When the frequency of the *sinusoidal* external force is slowly varied, the resonant steady-state response of the oscillator can occur at only one value of the driving frequency $\omega = \omega_{\text{res}}$, the resonant frequency of the oscillator. In other words, in the case of sinusoidal excitation there is only one resonance, and it occurs when the driving period T equals the natural period T_0 of the oscillator. However, in the case of the square-wave excitation, resonance occurs each time the driving period T is an odd-number multiple of the natural period T_0 of the oscillator, that is, when $T = (2n+1)T_0$, where $n = 0, 1, 2, \ldots$ Resonances, for which $n \geq 1$, occur when the frequency of one of the odd harmonics of the driving torque approaches the resonant frequency of the oscillator. Each resonance corresponds to a certain harmonic in the input spectrum.

Generalizing, we note that a linear oscillator with a sharp resonance curve (and given resonant frequency) appreciably responds only to a certain single harmonic

component of an arbitrarily complex external force. In this respect such an oscillator can be regarded as a spectral instrument, which selects a definite spectral component of an external action. That is, if we cause the natural frequency of oscillator to "sweep" through a range of frequencies, such oscillator responds resonantly each time its natural frequency coincides with one of the harmonic frequencies in the Fourier expansion of the external force. In other words, a sweep-frequency oscillator with a large quality factor provides us with a means by which a complex periodic input can be *physically* decomposed into its Fourier components.

The mathematical representation of the square-wave function in the right-hand side of Eq. (3) is not unique. The function can be represented as a sum of other functions in many different ways. That is, it is possible to express the external action either as a Fourier series of sine and cosine functions or as a series of other complete sets of functions. From the mathematical point of view, all such decompositions are equally valid. The usefulness of the Fourier decomposition in the case under consideration is associated with *physics.* It is related to the capability of a linear harmonic oscillator to perform this decomposition physically. When the phenomenon of resonance is used as a means of experimental investigation, only the Fourier representation of the analyzed complex process is adequate and expedient.

4. Forced oscillations as natural oscillations about the alternating equilibrium positions

Another way to obtain an analytic solution to the differential equation of motion (3) for steady-state oscillations forced by the square-wave external torque is based on viewing the steady-state motion as a sequence of free oscillations, which take place about an equilibrium position that periodically alternates between $+\phi_0$ and $-\phi_0$. For the first half-cycle, from $t = 0$ to $T/2$, the general form of the dependence of $\varphi(t)$ on t can be written as:

$$\varphi(t) = \phi_0 + Ae^{-\gamma t}\cos(\omega_1 t + \theta), \qquad (0,\ T/2), \qquad (8)$$

where $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of damped natural oscillations, and A and θ are arbitrary constants of integration determined by conditions at the beginning of the half-cycle. During the next half-cycle $(T/2,\ T)$ the time dependence of $\varphi(t)$ has the form:

$$\varphi(t) = -\phi_0 - Ae^{-\gamma(t-T/2)}\cos(\omega_1(t - T/2) + \theta), \qquad (T/2,\ T), \quad (9)$$

where the constants A and θ have the same values as they do in Eq. (8). These values follow from the fact that, in steady-state oscillations, the graph of time dependence during the second half-cycle must be the mirror image of the graph for the first half-cycle, shifted by $T/2$ along the time axis. This relationship is clearly seen in Fig. 3, where the graphs correspond to $T = 3T_0$.

The constants A and θ for any given values of T, ϕ_0, and γ can be calculated from the condition that during the instantaneous change in the position of the driving rod at $t = T/2$ the angular deflection and the angular velocity of the flywheel do not change. In other words, we should equate the right-hand sides of Eqs. (8) and (9) and their time derivatives at $t = T/2$. These conditions give us two simultaneous equations for A and θ. Solving the equations we find:

$$\tan\theta = -\frac{e^{-\gamma T/2}[\omega_1\sin(\omega_1 T/2) + \gamma\cos(\omega_1 T/2)] + \gamma}{e^{-\gamma T/2}[\omega_1\cos(\omega_1 T/2) - \gamma\sin(\omega_1 T/2)] + \omega_1} \qquad (10)$$
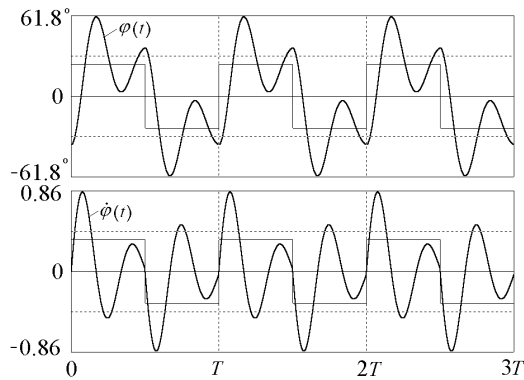
Figure 3. Graphs of the time dependence of the deflection angle and the angular velocity at resonant steady-state oscillations for $T = 3T_0$.

and

$$A = -\frac{2\phi_0}{e^{-\gamma T/2}\cos(\omega_1 T/2 + \theta) + \cos\theta}. \tag{11}$$

Equations (8)–(11) describe the steady-state motion only during the time interval from 0 to T. That is, if we substitute a value of t greater than T into these equations, they do not give the correct value for $\varphi(t)$. Nevertheless, we can find the value of $\varphi(t)$ for an arbitrary t by taking into account that $\varphi(t)$ is a periodic function of t: $\varphi(t+T) = \varphi(t)$. Thus, having obtained the graph of $\varphi(t)$ for the time interval $[0,\ T]$, we can simply translate the graph to the adjacent time intervals $[T,\ 2T]$, $[2T,\ 3T]$, and so on.
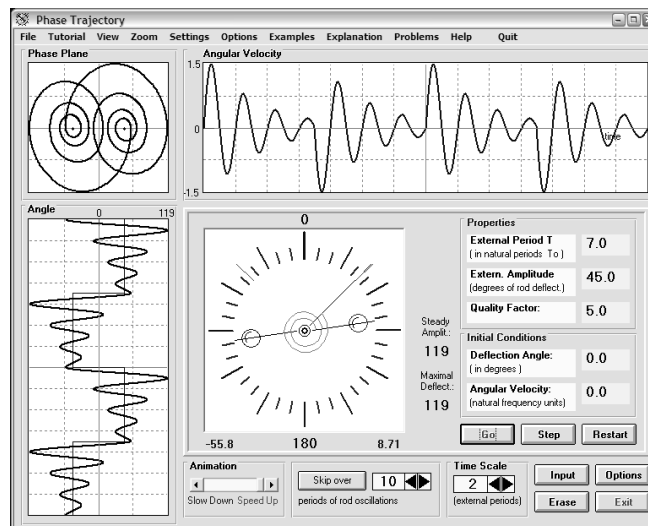

Figure 4. Damped oscillations about alternating displaced equilibrium positions at resonant steady-state oscillations for $T = 7T_0$.

The treatment of forced oscillations as natural oscillations about alternating equilibrium positions provides especially clear explanation of a rather complex behavior of the oscillator under the square-wave force whose period is considerably longer than the natural period. Figure 4 shows the screen image displayed by the computer program [4] simulating the steady-state forced oscillations at $T = 7T_0$ and relatively strong friction ($Q = 5$). The upper left-hand part of the screen shows the phase trajectory. The graph that displays the time dependence of the angular deflection is situated below the phase trajectory. The time axis of this graph is directed vertically down. Such an arrangement of the time-dependent graphs and the phase trajectory facilitates comparing the graphs of the angular position and velocity with the motion of the representing point along the phase trajectory. We see clearly how after each in turn abrupt displacement of the driving rod, the flywheel makes several natural oscillations of gradually diminishing amplitude about the new equilibrium position. These natural oscillations replace both abrupt fronts of each rectangular impulse distorting thus its shape from input to output.

5. Transient processes under the square-wave external torque

The above treatment of forced oscillations excited by a square-wave external torque as natural oscillations about alternating equilibrium positions enables us to clearly understand many characteristics of both steady-state oscillations and transient processes. In particular, it clarifies the physical reason for the resonant growth of the amplitude when the period of the driving force equals the natural period of the oscillator or some odd-number multiple of that period.

Suppose that before the external square-wave torque is applied, the oscillator has been at rest in its equilibrium position, $\varphi = 0$. When, at $t = 0$, the driving rod abruptly turns into a new position, ϕ_0, the flywheel, initially at rest, begins to execute damped natural oscillations about the new equilibrium position at ϕ_0 with the frequency $\omega_1 \approx \omega_0$. This oscillation begins with an initial velocity of zero. As long as the rod remains at ϕ_0, the time dependence of the angular displacement of the flywheel, $\varphi(t)$, is

$$\varphi(t) = \phi_0 - \phi_0 \exp(-\gamma t) \cos \omega_0 t.$$

That is, the flywheel, starting out with $\varphi = 0$ at $t = 0$, passes through the equilibrium position $\varphi = \phi_0$ when $\omega_0 t = \pi/2$, and reaches its extreme deflection of nearly $\varphi = 2\phi_0$ at $\omega_0 t = \pi$. (Damping prevents it from quite reaching $\varphi = 2\phi_0$.) If $T = T_0$, the flywheel arrives at the extreme point $\varphi \approx 2\phi_0$ (and its angular velocity becomes zero) just at the moment $t = T/2$, when one half of the driving period has elapsed. At this moment the rod instantly moves to the new position $-\phi_0$, and the next half-cycle ($T/2$, T) of the natural oscillation starts again with an angular velocity of zero, but its initial angular displacement from the new central point is nearly $3\phi_0$. This value is nearly $2\phi_0$ greater than in the preceding half-cycle. It would be exactly $2\phi_0$ greater in the absence of friction, and the amplitude of oscillation would increase by the value $4\phi_0$ during each full cycle of the external force, provided the driving period equals the natural period of the oscillator (or some odd-number multiple of that period).

In a real system such an unlimited growth of the amplitude linearly with time is impossible because of friction. The growth of the amplitude is approximately linear

during the initial stage of the transient process. This resonant growth gradually decreases, and steady-state oscillations are eventually established, during which the increment of the amplitude occurring at every instantaneous displacement of the driving rod is nullified by an equal decrement caused by viscous friction during the intervals between successive jumps.

Such a process of gradual growth of the amplitude, which eventually results in oscillations of a constant amplitude, is depicted very clearly by the phase trajectory shown in left-hand upper corner of Fig. 5. Its first section is a portion of a spiral that starts at the origin of the phase plane and winds around a focus located at the point $(+\phi_0, \ 0)$. The next section is a segment of a similar spiral that winds around the symmetrical point $(-\phi_0, \ 0)$.
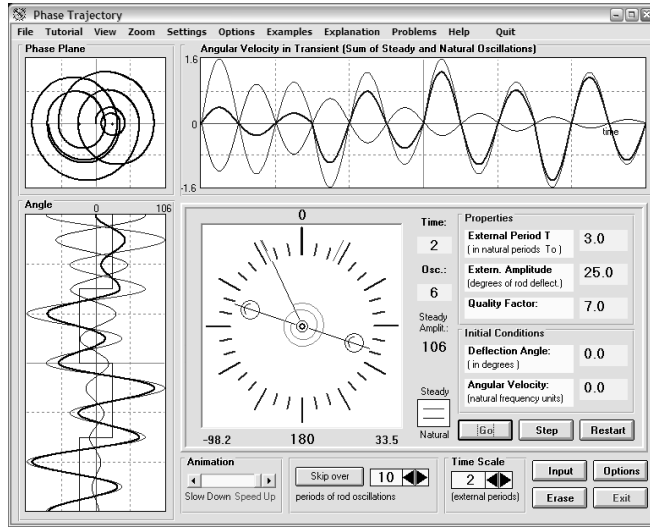

Figure 5. Graphs of the time dependence of the position angle and of the angular velocity, together with the phase diagram, for the transient process of excitation from equilibrium for resonance occurring at $T = 3T_0$.

If the period of excitation T equals an odd-numbered multiple of the natural period T_0, the transition from one spiral to the adjoining spiral (centered at the other focus) occurs at a maximal distance along φ-axis from the new focus. As a result, the new loop of the phase trajectory turns out to be larger than the preceding one. Such untwisting of the phase trajectory continues at a decreasing rate until the expansion of loops due to the alternation of the foci is nullified by their contraction caused by viscous friction. Eventually a closed phase trajectory is formed which corresponds to steady-state oscillations. This curve has a central symmetry about the origin of the phase plane. It consists of two branches each representing damped natural oscillations about one of the two alternating symmetrical equilibrium positions. For $T = 7T_0$ such a closed phase trajectory is shown in Fig. 4 (left-hand upper corner).

Any transient process in a linear system can be represented as a superposition of the periodic solution to Eq. (3) that describes the steady-state oscillations, and a solution of the corresponding homogeneous equation (with the right-hand side equal to zero) that describes the damped natural oscillations. The simulation program [4] displays such a decomposition of the transient process if the corresponding option is

chosen (see Fig. 5). One more example of such a decomposition in which the graph of damped natural oscillations is singled out especially clearly is given by Fig. 6 (curve *2* corresponds to the contribution of natural oscillations).
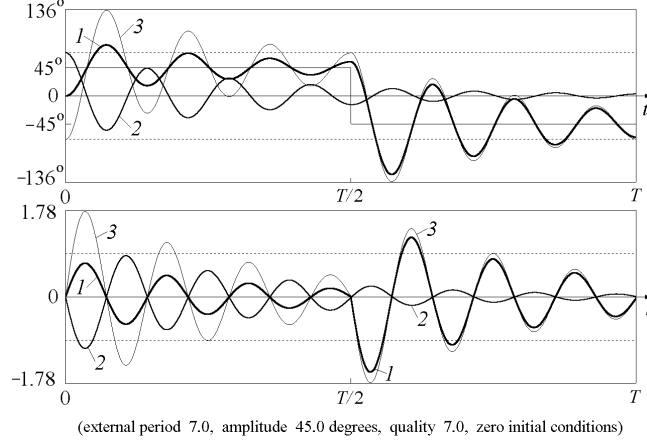

(external period 7.0, amplitude 45.0 degrees, quality 7.0, zero initial conditions)

Figure 6. Graphs of the angular position (upper part) and velocity (lower part) showing the decomposition of the transient (curve *1*) onto the damping natural oscillations (curve *2*) and the periodic steady-state oscillations (curve *3*).

6. Estimation of the amplitude of steady oscillations

Next we evaluate the maximal angular displacement, φ_m, attained in the steady-state oscillations. The value φ_m certainly can be found from exact Eqs. (8)–(11). However, such a calculation is rather complicated. Using the simple arguments suggested in the previous sections, we can avoid tedious calculations, at least for some special cases.

We consider first the main resonance in which the driving period equals the natural period: $T = T_0$. The closed phase trajectory for the steady-state oscillation consists in this case of a single loop intersecting the φ-axis at the extreme points $-\varphi_m$ and φ_m. The angular separations of these points from the equilibrium position at ϕ_0 equal $\varphi_m + \phi_0$ and $\varphi_m - \phi_0$ on the left and right sides of ϕ_0 respectively. The upper part of the phase trajectory is a half-loop of a spiral whose focus is at the point $+\phi_0$. While the representative point passes along this upper half-loop from $-\varphi_m$ to φ_m, the oscillator executes one half of a period of damped natural oscillation about the equilibrium position ϕ_0. When the oscillator reaches this extreme point, the equilibrium position switches to the focus $-\phi_0$, and the representative point then passes along the lower half-loop, thus closing the phase trajectory of the steady-state motion.

The relative decrease of the amplitude because of viscous friction during one half of the natural period ($t = T_0/2$) equals $\exp(-\gamma T_0/2)$. So the left and right extreme separations from ϕ_0 for the upper half-loop are related to one another through this exponential factor giving the frictional decay for a half-cycle:

$$(\varphi_m + \phi_0)\exp(-\gamma T_0/2) = \varphi_m - \phi_0. \qquad (12)$$

For the case in which $\gamma \ll \omega_0$, that is, $\gamma T_0 \ll 1$ (oscillator with relatively weak friction), we can assume $\exp(-\gamma T_0/2) \approx 1 - \gamma T_0/2$. Using this approximation in Eq. (12) and solving for φ_m, we obtain the desired estimate:

$$\varphi_m \approx \phi_0 \frac{2}{\gamma T_0/2} = \frac{4}{\pi} Q \, \phi_0. \tag{13}$$

The product of the damping constant γ and the natural period T_0 is expressed here in terms of the quality factor $Q = \omega_0/2\gamma$.

Equation (13) shows that for resonance induced by the fundamental harmonic of the square-wave external torque ($T = T_0$) the amplitude of steady-state oscillation is approximately Q times greater than the amplitude $(4/\pi)\phi_0$ of this harmonic in the square-wave motion of the rod. (See Eq. (5).) The same conclusion can be reached from a spectral approach to the treatment of stationary forced oscillations.

Through a similar (though more complicated) calculation we can evaluate the maximal displacement in steady-state oscillations for any of the higher resonances when the period of the square-wave external torque is an odd multiple of the natural period. When the driving period is an even-numbered multiple of T_0, the maximal displacement of the flywheel in steady-state forced oscillations cannot exceed $2\phi_0$. We can easily see this from the shape of the corresponding phase trajectory: each of its two symmetrical halves consists of an integral number of shrinking loops of a spiral winding around one of the foci ϕ_0 and $-\phi_0$. Figure 7 shows this kind of the phase trajectory and the time-dependent graphs for a special case in which $T = 4T_0$. For $T = 2T_0$, one complete cycle of the natural oscillation occurs while the equilibrium position is displaced to one side. In the absence of friction both closed loops of the steady-state phase trajectory meet at the origin of the phase plane, and the magnitude φ_m of the maximal displacement on each side of the zero point just equals $2\phi_0$. Friction causes the loops to shrink, and the maximal displacement φ_m becomes somewhat less than $2\phi_0$.

For high frequencies of the external force, when the square-wave period T of the driving rod is very short compared to the natural period T_0 of the oscillator, in steady-state motion the flywheel executes only small vibrations about the mid-point $\varphi = 0$. The period of these vibrations is the same as that of the driving rod. They occur in the opposite phase with respect to the rod motion, and their amplitude is small compared to the amplitude ϕ_0 of the rod.

Since the flywheel moves little while the position of the rod is fixed at either ϕ_0 or $-\phi_0$, we can consider the torque of the spring exerted on the flywheel as nearly constant during the intervals between successive jumps of the rod. Therefore the graph of the angular velocity consists of nearly rectilinear segments. Each segment corresponds to the rotation of the flywheel in one direction with a uniform angular acceleration $\omega_0^2 \phi_0$ caused by the constant torque of the strained spring. After each succeeding jump the acceleration changes sign, remaining nearly the same in magnitude. In the graph of the angular velocity the straight segments join to form a saw-toothed pattern of isosceles triangles. The corresponding graph of the angular displacement is formed by adjoining parabolic segments alternating after each half of the driving period. We can easily find the height of these segments, that is, the maximal displacement φ_m, by calculating the angular path of the rotor as it moves with a constant angular acceleration $\omega_0^2 \phi_0 = 4\pi^2 \phi_0/T_0^2$ from the zero point of the dial to φ_m during a quarter driving period $T/4$: $\varphi_m \approx (\pi^2/8)(T/T_0)^2 \phi_0$.
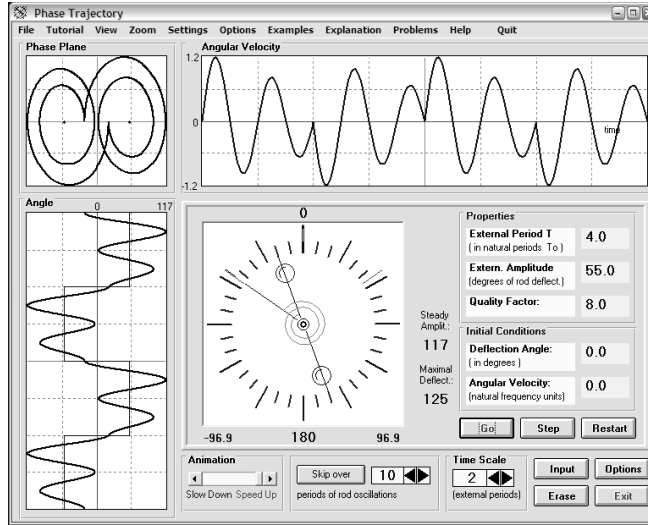
Figure 7. The graphs of the time dependence of the angular position and of the angular velocity together with the phase diagram for steady-state forced oscillations at $T = 4T_0$.

7. Energy transformations

The exchange of mechanical energy between the oscillator and the source of the external driving force occurs in the investigated system only at the moments when the driving rod jumps from one position to the other. During the intervals between such jumps, while the oscillator executes damped natural oscillations about any of the two displaced equilibrium positions, only an alternating partial conversion between the elastic potential energy of the strained spring and the kinetic energy of the flywheel occurs, accompanied by the gradual dissipation of mechanical energy because of friction.

A parabolic potential well corresponds to each of the two equilibrium positions of the oscillator (Fig. 8). When the flywheel is located at the angle φ from its central position, the corresponding potential energy of the spring is given by one of the two quadratic functions:

$$U(\varphi) = \frac{1}{2}k(\varphi \mp \phi_0)^2. \tag{14}$$

At an instantaneous jump of the driving rod from one position to the other the angular velocity of the massive flywheel and hence its kinetic energy do not change. An abrupt change occurs only in the value of the elastic potential energy of the spring. This causes the representative point to make an abrupt vertical transition from one of the parabolic potential wells to the other at a fixed value of the angle φ. During the interval before the next jump, while the oscillator executes damped natural oscillations about a displaced equilibrium position, the point which represents the total energy travels back and forth between the walls of the corresponding potential well, descending gradually toward the bottom because of energy losses caused by friction. This behavior is clearly seen in Fig. 8.
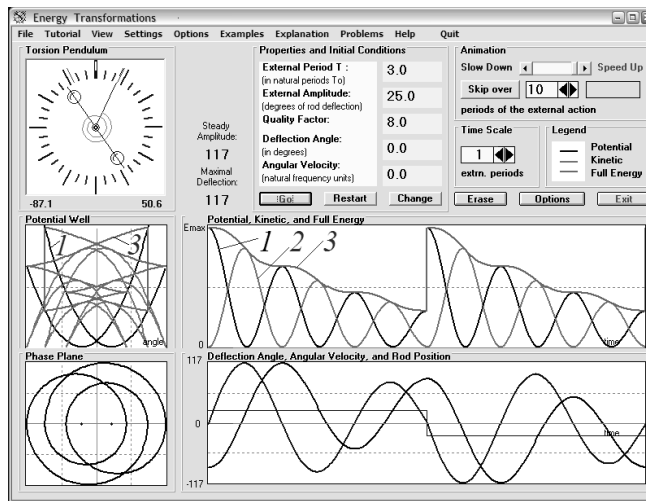
Figure 8. Energy transformations at steady-state oscillations with $T = 3T_0$. Curve *1* – potential energy, curve *2* – kinetic energy, curve *3* – total energy

It is important to note that in this simplified model of the physical system the deformation of the spiral spring is assumed to be quasistatic. In other words, we ignore the possibility that the spring vibrates as a system with distributed parameters, whose each portion has both elastic and inertial properties. For a light spring (attached to a comparatively massive flywheel) these vibrations are characterized by much higher frequencies than the frequencies of the torsional oscillations of the flywheel. These rapid vibrations of the spring quickly damp out.

The assumption concerning the quasistatic character of the spring deformation in the mechanical model, i.e., the assumption of possibility to neglect rapid vibrations of the spring as a system with distributed parameters, corresponds to the ordinary implicit assumption that the current in the analogous oscillatory circuit is quasistationary. According to this assumption, the momentary value of the current is the same along the whole circuit. The assumption is valid if the inductance and capacitance of the wires are negligible compared with the inductance of the coil and the capacitance of the capacitor respectively. In this case the oscillatory circuit can be treated as a system with lumped parameters (as a system with one degree of freedom), in which all the capacitance is concentrated in the capacitor and all the inductance is concentrated in the coil.

8. Concluding remarks

We have considered two different ways of determining the steady-state response of the linear oscillator to a non-harmonic driving force. The traditional approach based on the Fourier decomposition of the input driving force is certainly quite general because it is applicable to an arbitrary periodic excitation. The second method based on representing forced oscillations as some sequence of natural oscillations about displaced equilibrium positions can be used only for piecewise constant excitations. Nevertheless, this approach physically is much more obvious, and allows also to understand transient

processes. Combining both approaches in teaching students, we can hope to give them better understanding of this important topic.

The obvious, intuitive treatment of the transformation of a square-wave driving force (an input) into the steady-state oscillations of the mechanical spring oscillator (the output) described in this paper is equally valid for the transformation of an input square-wave voltage into the output oscillations of charge in the analogous electric circuit. Therefore behavior of this familiar mechanical system can help a student to better understand why and how an electromagnetic oscillatory LCR-circuit transfers the square-wave voltage from input to output with a distortion of its shape. Mechanical analogies allow a direct visualization and thus can be very useful in gaining an intuitive understanding of complex phenomena.

This simple mechanical system helps us to understand both the complex shape of the output oscillations and also their spectral composition. Only those harmonics of the input signal, which are close to the resonant frequency of the circuit, are noticeably present in the output voltage across the capacitor. In other words, such a resonant circuit selectively responds to different harmonic components of the input signal. The greater the quality factor Q, the sharper the resonance curve, and the finer the *selectivity* of the oscillatory circuit.

It is possible to easily vary the natural frequency $\omega_0 = 1/\sqrt{LC}$ of a resonant circuit by varying either the capacitance C or the inductance L. Such a tunable resonant circuit with high selectivity can serve as a spectral instrument that is able to accomplish the mathematical task of Fourier decomposition of a complex input signal on a physical level. The mechanical system described in this paper provides a clear and plain way to understand this possibility.

References

[1] Ch. Kittel, W. D. Knight, M. A. Ruderman, *Mechanics*, Berkeley Physics Course, vol. 1 (McGraw-Hill, New York, 1965)

[2] Ed. M. Purcell, *Electricity and Magnetism*, Berkeley Physics Course, vol. 2 (McGraw-Hill, New York, 1965)

[3] F. S. Crawford, *Waves*, Berkeley Physics Course, vol. 3 (McGraw-Hill, New York, 1966)

[4] Eugene Butikov, *Physics of Oscillations*. Part I. Simple Systems (Physics Academic Software, American Institute of Physics, New York, 1997). The software and the User's Manual can be ordered through the web: see `http://www.aip.org/pas`

[5] We note that in using the mentioned analogy it would be incorrect to exactly associate the electric potential energy of a charged capacitor with the elastic potential energy of a strained spring because the latter depends directly on the angle $(\varphi \pm \phi_0)$, while the energy of a capacitor depends directly on the charge q or on the corresponding voltage $V_C = q/C$ (*not* on the voltage $(V_C \pm V_0)$). In contrast to the spring oscillator, for which a jump in the position of the rod causes an abrupt change in the elastic potential energy of the spring, a jump of the input voltage across an electric circuit does not abruptly change the charge and the energy of a capacitor.

[6] A. Alenitsyn, E. Butikov, A. Kondratyev *Concise Handbook of Mathematics and Physics* (CRC Press, New York, 1997)