Rigid pendulum driven by a sinusoidal torque

Introduction

The simulation program “Rigid pendulum driven by a sinusoidal torque” deals with a classic nonlinear dynamical system which is interesting not only by itself but, more importantly, because it is isomorphic to many other physical systems (including rf-driven Josephson junctions and phase-locked voltage controlled oscillators) and therefore can be used to model various nonlinear systems and understand better their behavior. Mechanical analogies allow a direct visualization and thus can be very useful in gaining an intuitive understanding of complex phenomena.

Usually we expect that the pendulum will exhibit quite familiar behavior which agrees well with our intuition. However, despite the apparent simplicity, this familiar nonlinear system displays a rich variety of possible rather complex modes of motion which include various kinds of transient processes, single- and multiple-period stationary oscillations and complete revolutions, subharmonic and superharmonic resonance responses, bistability, intermittency, transient and stationary chaos. Most of these modes delight the eye and certainly challenge our physical intuition. Varying slowly the control parameters of the system (the driving frequency and amplitude, the damping factor), we can observe various kinds of bifurcations manifesting transitions of the pendulum between dramatically different modes of behavior.

A vast literature is devoted to the problem. Recently a large number of theoretical and experimental investigations have been carried out on the planar driven damped pendulum mainly in an effort to understand the nature of chaos in simple nonlinear systems [ ]. The seemingly simple situation of the forced pendulum occurred to be quite complex due to the subtle interplay between the periodic driving force and the natural modes of the pendulum. Strange as it may seem, an ordinary driven pendulum can exhibit an exceedingly complex response, which is even yet not fully explored. The suggested simulation program can illustrate all known modes of behavior of the sinusoidally driven pendulum, and serve also as a simple and useful means to discover new features of this really inexhaustible system.

The physical model

In the simulation program we consider an ordinary planar rigid pendulum, say, a weightless rigid rod with a massive bob at one end (a simple or mathematical pendulum), or any other massive body (a physical pendulum) that can rotate and swing about a horizontal axis in the uniform gravitational field. Being excited, the pendulum can rotate or swing in the vertical plane about the stable equilibrium position in which its center of mass is below the axis. These natural oscillations gradually damp due to friction whose braking torque is assumed in the model to be proportional to the angular velocity of the pendulum (viscous friction).

The period T_0 of infinitely small natural oscillations in the absence of friction is characteristic of the given pendulum and can serve as a convenient unit of time for the simulation. Momentary mechanical state of the pendulum is determined by its angular position φ, the angle of deflection from the vertical equilibrium position measured in radians (or degrees), and by the angular velocity $d\varphi/dt$ measured in the program in units of the natural angular frequency $\omega_0 = 2\pi/T_0$ of infinitely small oscillations of the pendulum. We assume that the pendulum is driven by an external sinusoidal torque with the frequency ω and some constant amplitude. The unit for this amplitude is chosen as follows. Imagine that some small constant (time-independent) external torque is exerted on the pendulum. The torque causes a displacement of the pendulum from the vertical. This angular displacement is proportional to the torque (for small enough values of torques), and can serve as a convenient measure of the torque. In the simulation, the amplitude of the external driving torque is expressed in units of this angle (in radians or degrees). This means that in the limit of low driving frequency (when ω tends to zero and the pendulum adiabatically follows the external torque), the steady-state forced oscillation of the pendulum will occur just with the amplitude of the driving torque measured in these angular units (provided the amplitude is small enough so that the static displacement is proportional to the torque).

The differential equation of motion used to simulate the damped driven pendulum in the program is of the form

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \sin\varphi = \omega_0^2\theta_0 \sin\omega t,$$

where ω_0 is the frequency of (undamped) natural oscillations in the low amplitude limit, ω is the driving frequency, γ is the damping factor, and θ_0 is the amplitude of the external driving torque measured in the units of angle, as explained above. When entering parameters of the system in the simulation program, we must express this amplitude in degrees of angle. We need not enter any value for the natural frequency ω_0 because this frequency is assumed as a convenient unit of the angular frequency (and of the angular velocity) in the simulation. Therefore the driving frequency ω must be expressed in these units for input, and we should enter a dimensionless value. For example, if we enter 1 for ω, the driving frequency will be equal to the frequency ω_0 of small natural oscillations of the pendulum. To measure the viscous damping, we use in the program instead of γ a more convenient dimensionless quantity Q – the quality factor that equals the ratio $\omega_0/2\gamma$.

Description of the program

The physical system

Executing the simulation, we can open different windows (panels) to display the motion of the system and represent its characteristics in more or less detail. Choosing “Physical system” from the main menu (or under the menu item “View” in any other window), we make the program display only a schematic image of the simulated system and the panel with the current values of parameters. The command button labeled “Start/Pause/Go” launches the simulation, or allows us to make pause or to continue the simulation. The button “Reset” retrieves the initial conditions. The button “Input” opens the panel that allows us to change the parameters of the system and/or the initial conditions.

The dial shows the angular deflection of the pendulum measured in degrees from the lower (stable) equilibrium position. When the pendulum swings, its extreme angular excursion on either side from the downward vertical position (in degrees) is marked near the image each time the pendulum changes the direction of motion. The speed of animation depends on the performance of your computer. You can vary this speed in some limits with the help of the scroll-bar labeled “Speed up” or “Slow down” the animation.

The time-dependent driving torque is conventionally shown by the green radial line that moves to and fro (swings) about the downward vertical line. The red radial lines on both sides show the limits (the amplitude values) of this driving torque measured in angular units. If we choose a low driving frequency (say, 0.1 – 0.2) and a small driving amplitude (say, 5 – 15 degrees), the pendulum, after the transient process is over, will swing slowly between these limits following adiabatically the green line that shows the variations of the external driving torque.

The time-dependent graphs

Choosing the item "Plots of oscillations" from the menu, we make the program display also the time-dependent graphs of the angle of deflection and of the angular velocity of the pendulum (and the graphs of the driving torque) alongside the image of the pendulum. The graphs are plotted simultaneously with the simulation of motion. The time scale (the number of driving periods displayed on the screen) can be changed by the user with the help of the spin-button "Time scale" (or by typing the desired number of periods directly into the corresponding box). If you use a graphic mode with a resolution higher than 800 x 600, you can expand the plots to make them look better by clicking the item "Zoom" in the menu. The program chooses the scale for the graphs (the maximum values of the angular position and angular velocity to be displayed on the graphs) automatically. You can set these scales manually if you wish with the help of the panel "Options" which can be opened by clicking the corresponding item in the menu. This panel allows also to choose a convenient mode for drawing the graphs: either to make a pause in the simulation and wait for further commands when all the space for the plots is used, or to continue the simulation drawing the new graphs over the old ones in the same frames (in the former scale), or to erase the old graphs (and rescale) before drawing the new ones.

The phase trajectory and Poincare mapping

For a real experimental work on your own, the most convenient choice is the item "Phase trajectory," which offers the screen with the image of the pendulum, the time-dependent graphs of the angular position and angular velocity, and the phase plane (φ, $d\varphi/dt$). Actually, the driven pendulum has a three-dimensional phase space spanned by φ, $d\varphi/dt$, and t, because the external torque exerted on the pendulum depends explicitly on time. The representative point which shows the changing mechanical state of the pendulum generates in this three-dimensional phase space a non-crossing trajectory that spirals around the time axis. But it is convenient just to plot the projection of this curve onto the phase subspace – the plane (φ, $d\varphi/dt$).

A standard technique in dealing with this three-dimensional phase space is to inspect the projection (φ, $d\varphi/dt$) of the representative point whenever the time t is a multiple of the driving period $T = 2\pi/\omega$. Such points on the phase plane (φ, $d\varphi/dt$) are called the Poincare sections. In case of a steady-state motion which repeats itself after each driving period, all the sections fall to the same fixed point of the phase plane (φ, $d\varphi/dt$). Thus, for a motion which is periodic at the driving frequency, only one point occurs repeatedly in this plane. Otherwise, the sections are scattered. During a transient that leads to the period-1 steady-state motion, the Poincare sections gradually condense to this fixed limiting point. If only a few points occur after the transient is over, then the motion is periodic with the period that equals the number of fixed points. If the sections do not condense to some finite set of fixed points, the motion never repeats and is chaotic. To display these Poincare sections during the simulation, we can check the corresponding check-box in the panel “Options.”

In the simulation program, a special arrangement of the time dependent graphs for $\varphi(t)$ and $d\varphi/dt$ is used on the computer screen when you choose the item “Phase plane” in the menu. This arrangement is especially convenient for comparing these graphs with the motion of the representative point along the phase trajectory in the plane (φ, $d\varphi/dt$). We note that the axes for $\varphi(t)$ have the same orientation and the same scale both on the phase plane (φ, $d\varphi/dt$) and on the time-dependent graph of $\varphi(t)$. That’s why the time axis on the graph $\varphi(t)$ has an unusual vertical orientation. The spin-button “Time scale” allows you to chose a convenient number of driving periods to be displayed on the screen. Clicking the menu item “Zoom,” you can improve the graphs if you use a screen mode with resolution higher than 800 x 600.

The panel “Phase trajectory” allows you to slightly vary parameters of the system (the driving amplitude and frequency, and the quality factor) directly during the simulation, without opening the panel “Settings.” To do this, you can use small spin-buttons near the boxes that show the values of corresponding parameters.

While working with the screen “Phase trajectory,” you can reach after some time a steady-state motion of the pendulum. (Certainly, this is possible only if the values of parameters are such that the motion is not chaotic.) When friction is weak, reaching the steady-state may require many driving periods. To save time, you can skip the animation for a number of driving periods which you can indicate in the corresponding box. When you click the command button “Skip the animation,” the program solves the equation of motion much faster without displaying the motion and plotting the graphs. Clicking then the “Go” button, you can observe the further motion of the pendulum.

Spectrum of oscillations

When a steady-state oscillation is reached, the current Poincare sections form a finite fixed set of points on the phase plane. If this occurs during the simulation, the menu item “Spectrum” becomes enabled. Choosing this item, you open a new screen which allows you to observe the spectrum of this periodic process. By clicking relevant command buttons, you can make the program draw the graphs of separate harmonics of this periodic process, and also the graph of their sum. You can compare the relative amplitudes and phases of different harmonics. Then, by clicking the “Go” button, you can compare the graphs of harmonics and of their sum with the graph obtained by the simulation of the process.

Nonlinear resonance curve

The simulation program offers a convenient way to observe the nonlinear resonance response of the pendulum as the drive frequency is smoothly varied from one side of the natural frequency through this frequency and to the other side, while the amplitude of the driving torque is held constant. The pendulum responds differently according to direction of the frequency variation – there is an associated hysteresis characterized by abrupt jumps in the amplitude and phase of the steady-state response. To simulate the pendulum behavior as the driving frequency varies and to obtain a resonant curve, we choose the item “Nonlinear resonance” in the main menu, or, from the “Phase trajectory” screen, choose the item “Frequency variation” in the falling down menu which appears if we click the menu item “View.” The panel appears which shows the image of the pendulum and the phase diagram, and two additional frames for plotting the graphs of the amplitude—frequency and phase—frequency characteristics of the pendulum.

Preparing the experiment, we can choose the range for the driving frequency variation, and the initial value of the frequency. We can use one of the two offered modes of the driving frequency variation: a uniform or a steady-state sweeping. If we choose the latter mode (with the help of the corresponding option button), each next in turn small increment (or decrement) of the driving frequency occurs only when the transient caused by the preceding step is over. There are command buttons that allow you to start or to stop the frequency sweeping, or to reverse its direction.

When you open the window or change the system parameters, the program draws two approximate theoretical curves in the relevant frames: the amplitude—frequency and phase—frequency characteristics (if the appropriate option is checked). For small driving amplitudes and relatively strong friction, the resonance curve looks much like a slightly distorted resonance peak of a linear damped oscillator. But for greater driving amplitudes and weak friction the resonance peak at its maximum is shifted to lower frequencies and acquires a shape typical for nonlinear systems with a “soft” restoring force. Above some critical value of the driving amplitude (for a given quality factor), the theoretical resonance curve becomes S—shaped with three solutions, only two of which are stable. This means that a hysteresis can develop in the experimental curve. When in the process of frequency sweeping an abrupt jump occurs from one slope of the inclined resonance peak to the other, not only the amplitude of the steady-state oscillations changes considerably, but their mode itself undergoes a dramatic change. The simulation of motion and the phase trajectory plotted simultaneously allow us to observe how this happens during a long transient that accompanies the amplitude jump.

Experiments that use the mode of steady-state sweeping of the driving frequency may require much time for obtaining the resonance curve, especially in cases of weak friction, when the transients last many driving periods. There is an option which allows you to save time: if you uncheck the check-box “Show the motion,” the frequency sweeping occurs much faster. While sweeping the driving frequency, you can check this box to observe the motion of the pendulum only for interesting intervals during which the jumps occur from one slope of the resonance curve to the other, and the character of motion changes radically.

The predefined examples

In order to reproduce on your own some specific mode of the pendulum’s behavior in the simulation experiment, you should enter the relevant values of the system parameters (the driving amplitude and frequency, and the quality factor), and suitable initial values of the angular position and angular velocity. For a fast overview of known modes, the simulation program offers you a predefined set of examples. These examples allow you to observe the chosen mode without the laborious business of finding and entering necessary parameters.

Clicking “Examples” in the menu, you open a panel with a list of predefined examples. When you select one of them, a brief description appears in the text-box below. To launch the simulation of an example, select it in the list and click the “Ok” button (or simply double-click the example in the list). Then the necessary values of all the parameters will be entered automatically, and you can immediately observe the corresponding simulation by clicking the “Start” button.

Creating new examples

The program provides two sets of predefined examples (basic set and extended set). You can choose one of them with the help of the corresponding option button in the panel “Examples.” You can create new additional sets of examples on your own. To do this, choose the third option “Custom set” offered by the same option button. Then the menu item “Edit” of this panel becomes enabled, and you can modify the existing set of examples by removing some items and adding new ones. You can also rearrange the set by moving examples up or down through the list. After the modified set of examples is finished, you can save it in a different file. Afterwards you can load and use it (or modify again) whenever you need.

To add an example to the list, first you should find the values of parameters that provide the desired motion, enter them in the panel “Settings,” and verify your choice by executing the simulation. Then you should open the panel “Examples” and choose the menu item “Edit.” By clicking the item “Create example” in the fall-down menu, you open the dialog box in which you type the name of the example (which will appear afterwards in the list of examples), and also some text (on your discretion) with a brief description of the created example. Then click “Ok” button to finish the work. Your example is added to the list. You can move it to an appropriate position in the list. After the new set of examples is completed, you can save it in a file in order to use afterwards. To do this, choose “Save as …” under the menu item “File.” A dialog panel appears which allows you to give a name to the new set of examples, and a name to the file in which this set will be saved. This name you will use afterwards to open your set of examples by using the dialog panel that opens when you click “Open examples …” in the menu “File.”