A virtual lab to explore nonlinear oscillations

Introduction

Various linear systems constitute the backbone of introductory physics because they are described by mathematically simple linear differential equations. In particular, the treatment of oscillations in traditional physics courses is restricted, as a rule, to various harmonic oscillatory systems, whose behavior can be almost completely investigated analytically by means of rather simple mathematics accessible for college and undergraduate university students. However, we should realize that such restrictions may lead students to a distorted conception of oscillatory physics as a whole. Detailed investigation of linear systems is certainly very important, but insufficient.

The periodically forced oscillator with harmonic potential gives only periodic motion of the same period in the steady-state response. Specifically, under sinusoidal driving force this steady-state motion is also purely sinusoidal – its spectrum consists of the single (principal) harmonic with the driving period. Various anharmonic potentials – corresponding to nonlinear restoring forces – can lead to a great variety of different modes of transient and steady-state responses, including subharmonic and superharmonic resonances, hysteretic transient and chaotic steady-state behavior. Chaos, which is a type of essentially unpredictable behavior exhibited by a variety nonlinear deterministic systems, has been a subject of intense interest during recent years. In order to observe chaotic behavior, however, the equation of motion must be nonlinear. Numerical simulation becomes then an essential tool to aid understanding of the phenomenon.

The physical system

A combined torsion spring pendulum with a non-balanced rotor (flywheel) whose equilibrium and potential well are produced both by the linear restoring torque of an elastic spiral spring and by the gravity, can serve as an ideal physical model for study of nonlinear oscillations. An attractive feature of using this model for academic purposes is the possibility of natural transition from a linear to nonlinear system.

The simulated pendulum (figure …) consists of a rigid rod which can rotate freely in the vertical plane (about a horizontal axis). An elastic spiral spring is attached to the rotor. The spring provides the restoring torque whose magnitude is proportional to the angular displacement of the rotor from the equilibrium position: $N_{\text{spr}} = -D\varphi$. The other end of the spring either is fixed immovably (for investigation of free, or natural, oscillations of the rotor) or is attached to the exciter – the lever that can rotate about the axis common with that of the rotor. The spring is unstrained (that is, the rotor is in the equilibrium position) when the rotor is parallel to the exciter. The zero point of the dial corresponds to the vertical orientation of the exciter.

Two massive weights are fixed to the rotor's rod at equal distances from the axis of rotation. If the weights have equal masses, the rotor is balanced – its center of mass lies on the axis. In this case the force of gravity is not influential, and the only restoring torque is created by the elastic spring. The torsion spring oscillator with the balanced rotor is a linear system. Being perturbed from rest, the rotor executes natural oscillations about the equilibrium position. With the balanced rotor, these oscillations are purely sinusoidal (harmonic). Their frequency ω_0 depends on the moment of inertia I of the rotor, and on the spring constant D:

$$\omega_0 = \sqrt{D/I}. \tag{1}$$

his frequency of natural oscillations occurring solely under the elastic restoring force is used further on as a convenient conventional unit of frequency, and the period of such oscillations $T_0 = 2\pi/\omega_0$ is used a natural unit of time appropriate for the investigated model.

The torsion spring oscillator becomes a nonlinear system if we make the masses of the weights unequal. For convenience, we consider that the distances of the weights from the axis, as well as their total mass, are hold constant – only the mass of one of the weights is decreased by transferring some its part to the other weight, so that the mass of the latter is increased by the same amount. Thus, when we perturb in this way the balance of the rotor, its moment of inertia remains the same. However, the unbalanced rotor is subjected to the additional torque created by the force of gravity. This additional torque is proportional to the *sine* of the angle of rotor's deflection from the vertical, just like for the ordinary pendulum. This makes the torsion oscillator with the unbalanced rotor a nonlinear system.

The external excitation of the pendulum is realized in the model by constrained back and forth periodic motion of the exciting lever within some limits on both sides of the vertical position. This obvious (and easily visualized on the computer screen) mode of excitation can be referred to as the kinematical excitation because the motion of some part of the system (the exciter) is given rather than the explicit dependence on time of the exerted external torque. We assume that the angle ψ formed by the exciter with the vertical line sinusoidally depends on time: $\psi(t) = \psi_{\max} \sin \omega t$.

The differential equation of the oscillator

When the rotor is turned through an angle φ from the vertical, the restoring torque exerted on the rotor by the spring equals $-D(\varphi - \psi)$, where ψ is the angle that indicates the momentary position of the exciter (relative the vertical line, which is assumed as the zero point of the dial). The torque created by the gravitational force equals $\Delta mga \sin \varphi$, where Δm is the difference between masses of the upper and the lower weights, g is the free fall acceleration, and a is the distance between the centers of the weights and the axis. Applying to the rotor with the moment of inertia I the law of rotation of a solid about an axis, we can write

$$I\ddot{\varphi} = -D(\varphi - \psi) + \Delta mga \sin\varphi. \qquad (2)$$

Dividing all the terms by I and introducing the following notation

$$\Omega = \sqrt{\Delta mga / I}\ ,$$

we obtain the following differential equation that describes the modeled system:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi - \Omega^2 \sin\varphi = \omega_0^2\psi_{\max} \sin\omega t.$$

Here the term is added that describes damping of oscillations by viscous friction (γ is the damping constant, related with the dimesionless quality factor Q by the equation $Q = \omega_0^2/2\gamma$).

The quantity Ω in the equation has the physical sense of the frequency of natural oscillations of the unbalanced rotor in the absence of the spring under the gravitational force, which can occur about the stable equilibrium position in which the heavier weight is below the axis. In the differential equation, we must consider Ω^2 to be positive, if the heavier weight is above the axis at $\varphi = 0$ (in the vertical position, when the spring is unstrained), and negative otherwise. The first possibility, in which gravity acts in opposition to the spring, is certainly more interesting for investigating nonlinear behavior because it provides a great variety of different modes.

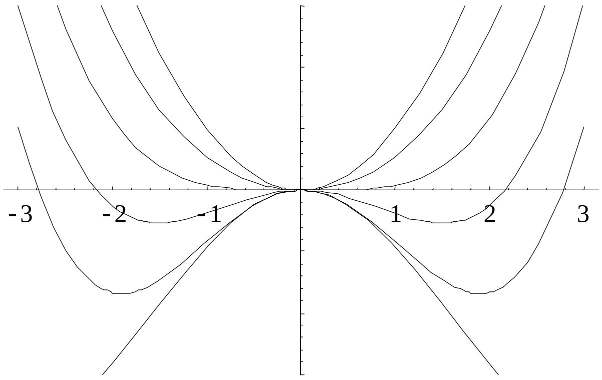
Free (natural) oscillations

In the absence of the external excitation, when the exciting rod is fixed in its middle (vertical) position, the oscillator, being perturbed from the state of rest, executes natural oscillations about the equilibrium position. With the balanced rotor, that is, if $\Delta m = 0$, these natural oscillations are purely harmonic because they occur only under the elastic torque, whose value is proportional to the angular displacement of the rotor. Potential energy of the strained spring is proportional to the square of the angular displacement, and the corresponding potential well is exactly parabolic. The frequency ω_0 of these oscillations is independent of the amplitude. This property of a linear oscillator is called *isochronism*.

In the general case, when the rotor is unbalanced (when $\Delta m \neq 0$ – one of the weights is heavier than the other), dependence of the potential energy on φ is more complicated:

$$U(\varphi) = \tfrac{1}{2} D\varphi^2 + \Delta mga\,(\cos\varphi - 1) = J\left[\tfrac{1}{2}\omega_0^2\varphi^2 + \Omega^2(\cos\varphi - 1)\right].$$

The graphs of the potential function $U(\varphi)$ for several values of $\Omega^2 > 0$ (the higher weight is heavier) are plotted in figure …



The upper curve corresponds to a balanced rotor (to the linear oscillator with $\Omega^2 = 0$), that is, this potential well is purely parabolic. The lower curve corresponds to the second term in $U(\varphi)$ with $\Omega^2 = \omega_0^2$, that is, to the (inverted) pendulum without a spring. Both curves are characterized by equal (but opposite) curvature at $\varphi = 0$. Therefore the potential well for $\Omega^2 = \omega_0^2$ has a flat bottom. The period of small oscillations in such a well tends to infinity as their amplitude tends to zero. The curves with $\Omega^2 < \omega_0^2$ describe nonlinear oscillators with "hardened" restoring force whose value increases with deflection from the equilibrium position faster than for the linear oscillator – the slopes of such a potential well rise more steeply than those of the corresponding parabolic well with the same curvature at the bottom. The period of natural oscillations in such potential wells depends on the amplitude – it decreases as the amplitude is increased.

To calculate approximately the period of such oscillations about the equilibrium position at $\varphi = 0$, we can make use of the equation of natural oscillations without friction:

$$\ddot{\varphi} + \omega_0^2\varphi - \Omega^2 \sin\varphi = 0.$$

The power expansion for sine ($\sin\varphi \approx \varphi - \varphi^3/6$) yields the following approximate nonlinear equation:

$$\ddot{\varphi} + (\omega_0^2 - \Omega^2)\varphi + \tfrac{1}{6}\Omega^2\varphi^3 = 0.$$

We can search its solution as a superposition of the principal harmonic with frequency ω and amplitude φ_{max}, and of the third harmonic with frequency 3ω and amplitude $\varepsilon\varphi_{\max}$:

$$\varphi(t) = \varphi_{\max}\sin\omega t + \varepsilon\varphi_{\max}\sin 3\omega t.$$

Substituting $\varphi(t)$ in the equation, we then equate to zero coefficients of $\sin\omega t$ and $\sin 3\omega t$, thus obtaining two equations, which determine the frequency ω and fractional amplitude ε of the third harmonic. Solving these equations, we find:

$$\omega^2 = \omega_0^2 - \Omega^2 + \tfrac{1}{8}\Omega^2\varphi_{\max}^2, \qquad \varepsilon = \frac{\Omega^2\varphi_{\max}^2}{192(\omega_0^2 - \Omega^2) + 21\Omega^2\varphi_{\max}^2}.$$

The simulation program allows us to easily verify this approximate value for the frequency of such nonharmonic natural oscillations. For example, if we take $\Omega^2 = 0.5$ and $\varphi_{\max} = 40^0$, the simulation gives for the period $T = 1.374T_0$, while the approximate theoretical value calculated with the help of this formula equals $1.373T_0$.

When the upper weight of the rotor is considerably greater than the other, so that $\Omega^2 > \omega_0^2$, the vertical equilibrium position becomes unstable, but two new stable equilibrium positions appear, which are symmetrically displaced on both sides of the relative maximum $\varphi = 0$. The angular positions of these displaced equilibrium points are determined by the following equation:

$$\omega_0^2\varphi = \Omega^2\sin\varphi,$$

whose solutions $\varphi = \pm\varphi_{\text{equil}}$ can be easily found by iterations. For example, if $\Omega^2 = 2\omega_0^2$, the equilibrium positions are found at $\varphi = \pm 108.6^0$. The equilibrium positions are at horizontal orientation of the rotor ($\varphi = \pm 90^0$) if $\Omega^2 = (\pi/2)\omega_0^2 = 1.5708\omega_0^2$.

The frequency of infinitely small natural oscillations about any of these displaced equilibrium positions is given by the following expression:

$$\omega^2 = \sqrt{\Omega^2\cos\varphi_{\text{equil}} - \omega_0^2}.$$

Small forced oscillations about the zero point