Parametric Excitation of a Linear Oscillator – Problems

1 Principal Parametric Resonance

1.1^* **Principal Resonance** $(n = 1)$ **in the Absence of Friction.**

(a) Input a moderate value of the depth m of modulation of the moment of inertia (about 10–15 percent). Choose the period of modulation T to be equal to one half the mean natural period of the oscillator. What kind of initial conditions ought you to enter in order to generate from the very beginning the fastest growth of the amplitude? Remember that at the initial moment, $t = 0$, the weights are suddenly moved away from one another, further from the axis of rotation.

(b) What initial conditions would lead at first to a fading away of oscillations that are already present? Using the plots of the oscillations, explain the physical reason for the increase or decrease in amplitude. Take into account the phase relationship between the natural oscillation of the rotor and the periodic changes in its moment of inertia. Why is it that some time later a phase relation is established that generates a growth in the amplitude?

(c) Try to understand the reasons that determine the lapse of time between the initial fading of the amplitude and its subsequent infinite growth.

1.2^* **The Amplitude Growth at the Principal Resonance without Friction.** (a) If the modulation is to generate the principal parametric resonance, what rule governs the growth of the amplitude when there is an initial deflection and an initial angular velocity of zero? Calculate the depth of modulation m that, in the absence of friction, generates a doubling of the amplitude after 10 cycles of the parametric modulation. Verify your result with a simulation experiment on the computer.

(b) What difference do you find in your observations of part (a) if you set the initial deflection to be opposite the deflection in part (a)?

1.3^* **The Threshold for Principal Resonance.**

(a) Choosing a moderate value for the modulation depth (say, $m = 0.15$), estimate the threshold (minimal) value of the quality factor $Q_{\min}$ that corresponds to stationary oscillations (i.e., to parametric regeneration) when the modulation is tuned to the principal resonance $(T = T_0/2)$.

(b) Make your calculated estimation of the threshold value $Q_{\min}$ more exact by using an experiment on the computer. Describe the character of the plots and of the phase trajectory under conditions of parametric regeneration and explain their features.

(c) Is the mode of stationary oscillations at the threshold (for $Q = Q_{\min}$) stable with respect to small deviations in the properties of the system? Is the mode stable with respect to small deviations in the initial conditions?

(d)** The threshold value of the quality factor for any given modulation depth m is absolutely minimal when the modulation is *exactly* tuned to resonance. For small values of m the principal resonance occurs when $T = T_0/2$. However, when m increases, the resonant value of the modulation period T departs from $T_0/2$. Find this resonant value of T for an arbitrarily large modulation depth m and estimate values of T for m = 15% and m = 40%.

1.4* **The Amplitude Growth over the Threshold.**

(a) For the case in which $T = T_0/2$ and $m = 15\%$, by what factor does the amplitude of oscillation increase during 10 cycles of parametric oscillation if $Q = 2Q_{\min}$? Does the answer depend on the initial conditions? Verify your answer with a simulation experiment.

(b) What is the amplitude of oscillation after the next 10 cycles of modulation? Why does friction not restrict the growth of the amplitude of parametrically excited oscillations?

1.5** **The Principal Interval of Parametric Resonance without Friction.**

(a) Calculate the values of the period of modulation T corresponding to the boundaries of the instability interval at a given modulation depth m (in the approximation $m \ll 1$) for the case when friction is absent.

(b) How does the width of the interval depend on the depth of modulation? Do the terms of second order influence the width of the interval?

1.6** **Oscillations at a Boundary of the Instability Interval.**

(a) Enter a value of the modulation period T which corresponds to one of the boundaries of the instability interval. Remember that at these boundaries stationary oscillations of constant amplitude are possible (parametric regeneration). If you then enter initial conditions arbitrarily, the amplitude of oscillations at first grows or decreases, and the shape of oscillations differs from that of the plots in figures 4 and 5 of the Manual. Why?

(b) Observe how the pattern of oscillations gradually approaches the shape that you should expect for the chosen boundary of the interval of instability. The oscillations preserve this shape for some time, but then the amplitude begins to grow or to decrease again, and the shape of oscillations changes again. Why?

1.7^* **The Initial Conditions for Steady Oscillations.**

(a) Enter the value of the period of modulation corresponding to the left boundary of the instability interval at a given value m of the modulation depth. Choose the absence of friction. Input some initial deflection. What value of the initial angular velocity ought you to enter for a given angular deflection in order that stationary oscillations of a constant amplitude occur from the beginning of the modulation?

(b) Verify your calculated approximate values of T for either boundary by simulating an experiment, and find more precise values. Explain the appearance of characteristic features of the plots and the phase trajectories of stationary oscillations corresponding to each boundary of the instability interval.

(c) For a given value of the initial displacement φ_0, and for the calculated value $\dot{\varphi}(0)$ of the initial angular velocity which provides stationary oscillations (at each of the boundaries of the interval of instability), calculate the amplitude of these oscillations. Verify the theoretical value by the experiment.

1.8^{**} **The Threshold of Excitation within the Instability Interval.**

(a) Choose a value T of the period of modulation somewhere between the limits of the interval of instability, e.g., approximately half way between the resonant value and one of the boundaries. Evaluate experimentally the growth of the amplitude in the absence of friction, and from your observations, calculate the threshold value of the quality factor $Q = Q_{\min}$ for parametric excitation at the given value T of the modulation period.

(b) Verify your result experimentally and use the experiment to find a more exact value of $Q_{\min}$. Compare the observed plots of these stationary oscillations with the plots of stationary (threshold) oscillations at exact tuning to resonance. What are the differences between the plots (and the phase trajectories) of stationary oscillations at the threshold within the interval of parametric excitation with friction, and the plots (and the phase trajectories) of stationary oscillations at the boundaries of the instability interval without friction?

(c) If the threshold is exceeded, why does the amplitude continue to increase indefinitely? In other words, why is friction unable to restrict the growth of the amplitude of parametrically excited oscillations?

(d) For small values of the modulation depth $m \ll 1$, calculate up to terms of second order in m the threshold value $Q = Q_{\min}$ of the quality factor for the period of modulation T lying somewhere within the interval of instability. Compare your theoretical result with the value which you have obtained experimentally in parts (a) and (b).

1.9^{***} **The Interval of Instability with Friction.**

(a) For some depth of modulation m, the frequency interval of parametric excitation shrinks because of friction and disappears as the quality factor reaches the threshold value. Let the quality factor Q be greater than the threshold value $Q_{\min}$. Find the values T_- and T_+ of the modulation period T which correspond to the boundaries of the instability interval for a given m and Q (in the approximation $m \ll 1$). Express these values in terms of m and $m_{\min}$, where $m_{\min} = \pi/(2Q)$ is the approximate threshold value of the modulation depth m for a given quality factor Q.

(b) In order to observe steady oscillations corresponding to these boundaries as soon as the simulation begins, you need to set the initial conditions properly. For a given value φ_0 of the initial deflection, and for each of the boundaries of the interval, what initial velocity produces steady oscillations from the very beginning? Verify your answer by simulating the experiment.

1.10 **Oscillations outside the Interval of Parametric Resonance.** For a given value of m, enter a value T of the modulation period lying somewhere outside the limits of the instability interval. Convince yourself that for any set of initial conditions the oscillations eventually fade away, even if the friction is very weak, and that the rotor comes to rest at the equilibrium position in spite of the forced periodic changes in its moment of inertia.

2 Parametric Resonances of High Orders

2.1* **The Third Parametric Resonance ($n = 3$) without Friction.**

(a) Examine the parametric excitation of the rotor for abrupt changes of its moment of inertia with the period $T \approx 3T_0/2$ (approximately one and a half times the natural period, or about three cycles of the parameter modulation during two cycles of natural oscillations). What initial conditions ensure the growth of the amplitude from the beginning of the modulation?

(b) What value m of the modulation depth in the absence of friction is necessary in order to double the initial oscillation during 15 cycles of the parameter modulation? After how many cycles does the amplitude double once more?

(c) For what initial conditions does the oscillation at first decrease? Why does this fading inevitably change after a while into an increase in the amplitude?

2.2* **The Threshold for the Third Resonance.**

(a) For small values m of the modulation depth, calculate the threshold value $Q_{\min}$ of the quality factor up to terms in the first order of m. How does this value depend on m? Compare your answer with the principal resonance, $n = 1$ (See Problem 1.3), and with the second resonance, $n = 2$ (See Problem 2.5). What might be a qualitative explanation for the difference?

(b) For $m \approx 30\%$ evaluate the minimal value $Q_{\min}$ of the quality factor for which parametric resonance of the order $n = 3$ is possible. Improve your theoretical estimate by simulating the experiment. Explain the observed shape of the angular velocity plot and the form of the phase trajectory of stationary oscillations at $Q = Q_{\min}$. What factor determines the amplitude of such oscillations?

2.3** **The Third Interval of Parametric Excitation.**

(a) Calculate the values of the modulation period T which, in the absence of friction, correspond to the boundaries of the third instability interval for a given modulation depth m (in the approximation $m \ll 1$). How does the width of the interval depend on the depth of modulation? Do the terms of the second order influence the width of the interval?

(b) What value of the initial angular velocity ought you to enter for a given initial deflection φ_0 in order to get stationary oscillations of a constant amplitude from the very beginning of the modulation on each boundary of the instability interval? Verify your calculated values experimentally.

What are the shapes of the phase trajectories that correspond to the left and right boundaries of this interval?

(c) Explore the width of the third interval of parametric excitation without friction at arbitrarily large values of the modulation depth m. Note how the interval moves to the left and gradually shrinks as m becomes greater. (See also figure 10 of the Manual.)

At m = 60% both boundaries of the interval coincide. (You may say also that they *intersect* at this value of m.) This coincidence means that for the corresponding value of the modulation period T you get steady oscillations for arbitrary initial conditions. What might be a physical explanation for this behavior?

Hint: What is the ratio of the natural periods for the maximal and minimal values of the moment of inertia for this value of the modulation depth?

2.4** **The Third Instability Interval with Friction.**

(a) At small values of m the third parametric resonance occurs at $T = 3T_0/2$. However, with the growth of m the resonant value of the modulation period T departs from $3T_0/2$. Find an analytical expression for this resonant value of T (for an arbitrarily large modulation depth m) and make a numerical estimate for m = 15% and m = 40%.

(b) How does friction reduce the width of the third interval of parametric excitation? For a small depth of modulation $m \ll 1$, calculate approximate values of the period of modulation T which correspond to the boundaries of the interval for a given value of the quality factor Q. Express the results in terms of m and the threshold value $m_{\min} = 3\pi/(2Q)$ (see Problem 2.2) for the given Q-value.

2.5** **Parametric Resonance of the Second Order** $(n = 2)$.

(a) Choosing a moderate value of the modulation depth ($m < 20\%$), excite parametric resonance of the order $n = 2$ (for which the period of modulation is approximately equal to the average natural period). Why does the growth in amplitude occur much more slowly in this case than it does for the principal resonance, and even than it does for the resonance of the order $n = 3$ (for the same value m of the modulation depth)? Explain the observed shape of the phase diagram for $n = 2$. Try to determine experimentally the threshold value of the modulation depth for a given value of the quality factor (say, $Q = 15$).

(b) For small values of the modulation depth $m \ll 1$, try to calculate the threshold value of the quality factor Q_{min}. (You need to keep the terms of the second order in m). How does the threshold value of Q depend on m? Compare your calculated value with the threshold of principal resonance and of the third resonance. Explain the difference qualitatively. Compare also the theoretical threshold value with your experimental result of part (a).

2.6** **The Second Interval of Parametric Excitation.**

(a) For small values of the modulation depth $m \ll 1$, calculate the width of the interval (you need to keep the terms to the second order of m). How does the width depend on m?

(b) Excite and experimentally examine stationary oscillations without friction which correspond to the boundaries of the second instability interval (near the resonance for $n = 2$). For small values of the modulation depth $m \ll 1$, why is this interval considerably narrower than the interval for resonance of a higher order $n = 3$?

(c) Why do two different phase trajectories correspond to each boundary of the interval? What is the difference between the two stationary oscillations that correspond to the same boundary? How can each one of them be excited? What initial conditions ensure steady oscillations from the beginning of modulation?

2.7*** **The Second Interval of Parametric Excitation with Friction.**

How does friction influence the width of the second interval of parametric excitation? For a small depth of modulation $m \ll 1$, calculate approximate values of the period of modulation T which correspond to the boundaries of the interval for a given value of the quality factor. Write down the results in terms of m and the threshold value Q_{min} (see Problem 2.5) for the given value of Q.