# Free Oscillations and Rotations of a Rigid Pendulum – Problems

## Summary of the Principal Formulas

The differential equation of motion of a rigid pendulum is:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \sin\varphi = 0,$$

where $\omega_0$ is the frequency of small free oscillations:

$$\omega_0^2 = mga/J = g/l; \qquad l = J/ma.$$

Here $m$ is the mass of the pendulum, $a$ is the distance between the horizontal axis of rotation (the point of suspension) and the center of mass, $J$ is the moment of inertia about the same axis, $l$ is the reduced length of the physical pendulum, and $g$ is the acceleration of gravity.

The equation of a phase trajectory in the absence of friction is:

$$\frac{\dot{\varphi}^2}{\omega_0^2} + 2(1 - \cos\varphi) = \frac{E}{E_0},$$

where $E$ is the total energy, and

$$E_0 = \frac{1}{2}J\omega_0^2 = \frac{1}{2}mga = \frac{1}{4}(E_{\text{pot}})_{\text{max}}.$$

Here $(E_{\text{pot}})_{\text{max}} = 2mga$ is the maximal possible value of the potential energy of the pendulum, which is its potential energy when it is in the inverted vertical position.

The equation of the separatrix in the phase plane is:

$$\dot{\varphi} = \pm 2\omega_0 \cos(\varphi/2).$$

The angular deflection and angular velocity for the motion of the pendulum which generates the separatrix in the phase plane are:

$$\varphi(t) = \pi - 4\arctan(e^{-\omega_0 t}), \qquad \dot{\varphi}(t) = \pm\frac{2\omega_0}{\cosh(\omega_0 t)} = \pm\frac{4\omega_0}{(e^{\omega_0 t} + e^{-\omega_0 t})}.$$

# 1 Small Oscillations of the Pendulum

At small angles of deflection, when $\sin\varphi \approx \varphi$, the restoring torque of the force of gravity is approximately proportional to the angle of deflection from the position of stable equilibrium, and the pendulum behaves like a linear oscillator. In the absence of friction it executes simple harmonic motion. In the presence of weak friction, its motion can be considered as a nearly harmonic oscillation with a slowly decreasing amplitude.

1.1 **The Amplitude, Phase Trajectory, and Energy of Small Oscillations.** Select the case of no friction and use initial conditions which produce oscillations of a small amplitude. For instance, let the initial deflection be 30° and the initial velocity be zero. In this case the amplitude will be 30°.

(a) What is the maximal value of the angular velocity in these oscillations?

(b) What initial angular velocity should you give the pendulum initially in the equilibrium position ($\varphi(0) = 0$) in order to excite oscillations of the same amplitude (of 30°)? Calculate the required value of the initial velocity assuming that small oscillations of the pendulum are approximately sinusiodal. Verify your answer with an experiment. Remember that the initial angular velocity you enter must be expressed in units of the frequency $\omega_0$ of small oscillations. What is the difference between these oscillations and oscillations excited by an initial deflection? Improve the required value of the initial angular velocity by using the law of energy conservation. Verify the improved value with an experiment.

(c) Convince yourself that at small amplitudes the graphs of the angle of deflection versus time and of the angular velocity versus time have shapes which are close to that of a sine curve. Convince yourself also, that oscillations of the velocity lead in phase the oscillations of the angular displacement by a quarter of a period.

Compare the graphs of time dependence of the deflection angle and of the angular velocity with the motion of the representative point along the phase trajectory. What is the form of the phase trajectory for small oscillations? With what scale along the ordinate axis of the phase plane is the phase trajectory approximately a circle?

(d) What can you say about the time dependence of the kinetic and potential energies of the pendulum at small amplitudes? Prove that the time average values of kinetic and potential energy are approximately equal. If the amplitude equals 30 degrees, what is the ratio of total energy $E$ to the maximal possible value of potential energy $E/E_{\max}$?

1.2* **Period of Small Oscillations.**

For graphs of the time dependencies of the angle of deflection and of the angular velocity, the scale shown on the time axis is in the appropriate units for a given pendulum, namely in units of $T_0 = 2\pi/\omega_0 = 2\pi\sqrt{l/g}$), which is the period of small oscillations of the pendulum. That is, the duration between hatch marks on the time axis is $T_0$.

(a) Note that at small but finite amplitudes (say about 30°), the period of oscillations is a bit longer than $T_0$. You can make this observation either from the curves plotted on the screen or from readings of the timer. In the latter case, you can stop the simulation by clicking the START / STOP button at the moment when the pendulum completes a whole number of cycles. As a convenience in taking further readings, you may set the timer to zero during a pause in the simulation by clicking the RESET TIMER button. Try to measure the period (in units of $T_0$) for several moderate values of the amplitude.

(b) In performing precise measurements of the period in the simulation experiments, which instants are better for starting and stopping the timer: when the pendulum passes through the equilibrium position or when it reaches the points of its greatest deflection? Explain your answer. Keep in mind that your goal is to measure the period with maximal possible precision.

(c) Compare the measurement of the period $T$ for a given amplitude $\varphi_0$ obtained from the simulation experiment with the value given by the theoretical approximation:

$$T = T_0(1 + \varphi_0^2/16), \tag{1}$$

in which the amplitude $\varphi_0$ is expressed in radians. Determine the maximal value of $\varphi_0$ for which Eq. (1) gives the value of $T$ to within one percent. Find the error the formula yields for an amplitude of 45 degrees.

1.3 **Damping of Small Oscillations.**

(a) Prove theoretically that weak viscous friction causes exponential damping of small free oscillations. At what value of the quality factor $Q$ does the amplitude halve during four complete cycles? Input the calculated value of $Q$ and verify it in a simulation experiment on the computer.

(b) Convince yourself that for large viscous friction for which the quality factor $Q$ is less than the critical value of 0.5, a disturbed pendulum returns to the equilibrium position without swinging. What is the principal qualitative difference of the phase trajectories for the cases of weak and strong damping?

# 2 Oscillations with Large Amplitudes

2.1 **Comparison of the Pendulum with a Linear Oscillator.** For large angular displacements from the equilibrium position, the nonlinearity of the dependence on the angle $\varphi$ of the restoring gravitational torque is more apparent. Because $\sin\varphi < \varphi$, the increase in the restoring torque with increasing angular deflection is not as large for a pendulum as it is for a linear oscillator. Therefore, a pendulum is referred to as a nonlinear oscillatory system with a "soft" restoring force.

(a) How do the differences between a pendulum and a linear oscillator reveal themselves in graphs of the time dependence of the angular deflection and the angular velocity? How do the differences reveal themselves in the phase trajectory? Give a qualitative physical explanation for the differences.

(b) What are the differences between the pendulum and a linear oscillator with respect to energy transformations? Compare the phase trajectory with the graph of potential energy versus deflection angle. The placing of the graphs on the computer screen (when you check the item "Phase Diagram") is especially convenient for such comparison. Pay special attention to the position of the extreme points on the phase trajectory and on the potential well of the pendulum. For given initial conditions $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \Omega$, what are the values of the potential energy and kinetic energy of the pendulum at the extreme points and at the equilibrium position?

2.2* **Oscillations with Large Amplitudes.**

(a) Study experimentally large oscillations of the pendulum in the absence of friction. Note the exact periodicity of these clearly non-sinusoidal oscillations of the dynamical variables in the conservative system.

When the amplitude exceeds 90$^\circ$, the graph of angular velocity versus time is nearly a saw-tooth with equilateral triangular teeth. Explain this shape.

The shape of a tooth in the corresponding graph of the angular deflection in this case is close to a parabola, in contrast to the sinusoidally shaped tooth associated with oscillations with small amplitudes. Explain this parabolic shape. Note the increase in the period with increasing amplitude. (Hatch marks on the time axis are separated by $T_0$, the period of small oscillations.)

(b) Note how the closed phase trajectories of the oscillating pendulum are stretched horizontally as the energy of the pendulum increases. Explain why these phase trajectories are different from the elliptical phase trajectories of a linear oscillator. To do so, use the shapes of the parabolic potential well of a linear oscillator and the sinusoidal potential well of the pendulum. Assume the curvature near the bottom to be the same for both potential wells: the period $T_0$ of small oscillations of the pendulum should be equal to the period of the linear oscillator. Remember that the latter period is independent of the energy.

Explain the increase of the period of the pendulum with increasing amplitude, comparing its potential well with that of a linear oscillator.

(c) At large amplitudes the pendulum passes rapidly through the vicinity of the equilibrium position (through the sinusoidal bottom of the potential well) and slowly climbs up the sinusoidal crest of the well, along its nearly horizontal upper slopes; then it slowly descends from them. So on the average the pendulum remains at large deflections longer than does a linear oscillator, whose parabolic potential well has steadily increasing slopes. Use the shapes of these potential wells to explain why, during a cycle, the time average values of the potential and kinetic energies of a pendulum are not equal to one another while those of the linear oscillator are.

(d)* Study carefully the interesting case of oscillations with an amplitude near 180°. Set the initial deflection to be 179.999°, and the initial velocity to be zero. After remaining near one side of the inverted position for a long time, the pendulum, rapidly passing through the bottom of its path, remains for a long time again near the other side of the inverted position.

Compare the time during which the pendulum covers almost all its circular path (except a small vicinity of the extreme positions) with the period of small free oscillations of the pendulum. In other words, estimate the duration of a solitary impulse on the graph of angular velocity versus time. Or, equivalently, estimate the width of the nearly vertical portion of the nearly rectangular saw-tooth graph of the angular deflection versus time.

(e)* Try to discover what factor determines the width of this nearly rectangular tooth of the graph $\varphi(t)$, or, equivalently, what factor determines the time interval between successive impulses in the graph of angular velocity versus time. That is, try to discover the physical cause that determines the complete period of these extraordinary oscillations of the pendulum. (Hint: set the initial deflection of the pendulum at the successive values 179.999°, 179.990°, and 179.900°, each with an initial velocity of zero.

(f)** Try to evaluate theoretically the time interval needed for the pendulum to reach the extreme deflection of 179.999 degrees at excitation from rest in the lower stable equilibrium position.

Use your results to estimate the period of oscillations with the amplitude 179.999 degrees. Compare your estimation of the period with the value of $T$ obtained in the simulation experiment.

(g) Note the character of energy transformations in the motion considered above. Total energy $E$ in this motion nearly equals the height $2mga$ of the potential barrier. It is the value of the potential energy of the pendulum in the inverted position ($\varphi = \pm\pi$). Since the pendulum spends most of its period near the inverted position (because the pendulum moves and accelerates very slowly while in the vicinity of the inverted position), the time averaged value of its potential energy, taken over a complete oscillation, is much greater than the mean value of its kinetic energy. In this case potential energy is converted into kinetic energy only for the short time during which the pendulum makes a rapid turn passing through the lower equilibrium position of minimal potential energy. Try to evaulate (to an order of magnitude) the ratio of the values of the potential energy to the kinetic energy, averaged over a period, during oscillations with an amplitude of 179.99°.

2.3* **Motion along the Separatrix.**

(a) When you set the initial deflection to be almost 180 degrees and the initial velocity to be zero, the phase trajectory of the resulting motion nearly coincides with the separatrix $\dot{\varphi} = \pm 2\omega_0 \cos(\varphi/2)$. The point representing the mechanical state of the pendulum in the phase plane passes rapidly along the lower branch of the separatrix, remains for a long time at the left saddle point $(-\pi,\ 0)$, and then returns along the upper branch of the separatrix. What initial conditions should you choose in order to make the representative point move first along the upper branch of the separatrix and then along the lower one?

(b) What value of the initial angular velocity $\Omega$ (in units of $\omega_0$) must be initially given to the pendulum in its lower equilibrium position in order to make the representative point in the phase plane move along the separatrix? What value of the initial angular velocity should you input if the pendulum is to be initially deflected from the equilibrium position by an angle of 60°? 90°? $-90$°? 120°? Verify your answers with simulation experiments.

(c) For the limiting motion along the separatrix, calculate the time interval $\tau$ during which kinetic energy of the pendulum is greater than its potential energy. Or, which is the same, for the pendulum making its circular path from one side of the inverted position to the other, find the lapse of time between the two instants at which the pendulum passes through the horizontal positions on either side of the lower equilibrium position. Express this time interval in units of the period $T_0$ of small oscillations. Verify your calculated value by the experiment on the computer.

2.4 **Large Oscillations with Friction.**

(a) Examine the influence of viscous friction on oscillations of large amplitude. Begin with rather weak friction ($Q \approx 20$). Note the gradual changes in the pattern of the graphs as friction slowly decreases the mechanical energy and the amplitude of the pendulum. In particular note how the initial triangular saw-tooth curve of angular velocity with its sharp nearly rectilinear teeth, as well as the initial curve of angular deflection with smooth parabolic crests, both evolve into the sinusoidal curves characteristic of the simple harmonic oscillator.

(b) Under the influence of viscous friction, the topologies of the phase trajectories of a pendulum change. Instead of closed curves corresponding to exactly periodic oscillations of a conservative pendulum, you see twisting spirals making an infinite number of gradually shrinking loops around the focus at the origin of the phase plane. Note how the form of the loops changes when they recede from the separatrix. Give a qualitative explanation for the observed changes. (For a linear oscillator experiencing viscous friction, the shrinking loops of the phase curve remain similar as the curve approaches the origin.)

(c) Using the graphs in the panel “Phase diagram” of program, note how the rate at which energy is dissipated depends upon the position of the representative point in the potential well. At which part of a cycle does the rate of energy dissipation reach a maximum? Explain your answer.

(d)** Using the law of energy conservation, calculate the minimal value of the initial velocity which the pendulum must be given in the lower equilibrium position in order to reach the inverted position, for the case in which there is no friction and for the case in which the quality factor $Q = 20$. What must be the initial velocity in order to reach the inverted position if the pendulum is initially deflected by the angle 60 degrees? By 90 degrees?

# 3 The Rotating Pendulum

A pendulum makes a full revolution if its total energy exceeds the value $2mga$, the maximal value possible for its potential energy. The influence of the gravitational force makes this rotation in the vertical plane nonuniform: the angular velocity is a maximum (in the absence of friction) each time the pendulum passes through the lower, stable, equilibrium position and a minimum each time the pendulum passes through the upper, unstable, equilibrium position.

3.1 **The Angular Velocity at Revolutions.**

(a) Select the case of the absence of friction. Calculate the minimal initial angular velocity needed to obtain a full revolution of the pendulum when it is initially at the lower equilibrium position. Note the character of the graph of angular velocity versus time: As the pendulum revolves, its angular velocity changes periodically (that is, the angular velocity oscillates in time), but the sign of the angular velocity does not change (that is, the curve does not intersect the time axis).

(b)* How does the period of these oscillations change if the initial angular velocity is increased? Calculate the minimal value of the oscillating angular velocity for a given value of the initial angular velocity. Find the asymptotic dependence of the period of rotation on the initial angular velocity $T(\Omega)$, valid for the values of total energy $E$ which are much greater than the potential energy of the inverted pendulum ($E \gg 2mga$).

(c) What initial conditions must be entered in order to obtain a phase trajectory located above the separatrix in the phase plane? Located below the separatrix? Coinciding with the upper or lower branch of the separatrix?

3.2* **The Period of Revolutions and Oscillations.**

(a) It is especially interesting to compare the period of rotation with the period of oscillation of the conservative pendulum whose total energy $E$ is close to the maximal possible value of the potential energy $E_{\max} = 2mga$. In this case, the phase trajectories lie in the vicinity of the separatrix. Using the simulation experiment, measure the period for two values of the total energy $E$ that slightly differ from $E_{\max}$ by equal amounts on either side of $E_{\max}$. For example, first let $E/E_{\max} = 0.999$ and then let $E/E_{\max} = 1.001$. If you excite the motion of the pendulum from the lower equilibrium position, what values of the initial velocity should you enter for imparting such energies to the pendulum? Remember that you should enter the initial velocity in units of natural frequency $\omega_0$ of infinitely small natural oscillations.

(b) What is the ratio of the periods you have measured in these two cases? How can you explain this ratio?

(c) When the total energy $E$ of the pendulum is greater than the height $E_{\max} = 2mga$ of the potential barrier, the period of rotation $T$ rapidly decreases as the energy is increased. The period tends to zero with the growth of the energy. What is the asymptotic behavior of $T(E)$ when $E$ tends to infinity?

3.3$^*$ **Rotation of the Pendulum with Friction.**

(a) Examine experimentally the rotation of the pendulum in the presence of weak viscous friction. Note the gradual approach of the phase trajectory to the separatrix. What is the value of the total energy of the pendulum at the moment when the phase trajectory crosses the separatrix? Note that before the crossing (while the pendulum is executing complete revolutions), the kinetic energy and the angular velocity of the pendulum are never zero.

(b)$^{**}$ Using the law of energy conservation, evaluate the minimal value of the initial velocity needed to obtain a complete revolution of the pendulum when it is initially in the lower position if the quality factor $Q = 20$. What value of the initial velocity is needed to obtain two revolutions of the pendulum? Verify your result in a simulation experiment. Try to improve the approximate theoretical value of the required initial velocity by trial end error.