Torsion Spring Oscillator with Dry Friction – Problems

Summary of the Principal Formulas

The differential equation of motion of an oscillator acted upon by dry friction:

$$J\ddot{\varphi} = -D(\varphi + \varphi_{\mathrm{m}}) \qquad \text{for} \qquad \dot{\varphi} > 0,$$
$$J\ddot{\varphi} = -D(\varphi - \varphi_{\mathrm{m}}) \qquad \text{for} \qquad \dot{\varphi} < 0,$$

where φ_m is the angle corresponding to the boundaries of the dead zone. If in addition, viscous friction is present, a term proportional to the angular velocity is also present:

$$\ddot{\varphi} = -\omega_0^2(\varphi + \varphi_{\mathrm{m}}) - 2\gamma\dot{\varphi} \qquad \text{for} \qquad \dot{\varphi} > 0,$$
$$\ddot{\varphi} = -\omega_0^2(\varphi - \varphi_{\mathrm{m}}) - 2\gamma\dot{\varphi} \qquad \text{for} \qquad \dot{\varphi} < 0,$$

where ω_0 is the natural frequency of oscillations in the absence of friction:

$$\omega_0^2 = \frac{D}{J}.$$

The damping factor γ that characterizes the viscous friction is related to the quality factor Q by the equation:

$$Q = \frac{\omega_0}{2\gamma}.$$

The boundary value of the amplitude that delimits the two cases in which the effects either of viscous friction or of dry friction predominate:

$$a = \frac{4\varphi_{\mathrm{m}}}{\gamma T} = \frac{4}{\pi}\varphi_{\mathrm{m}} Q \approx \varphi_{\mathrm{m}} Q.$$

1 Damping Caused by Dry Friction

The strength of dry friction in the system is characterized by the width of the dead zone. This interval is defined in the program when you input the value of the angle φ_m which sets the limits of the dead zone on both sides of the middle position at which the spring is unstrained. Total width of this dead zone is $2\varphi_{\mathrm{m}}$. The value of φ_{m} must be expressed in degrees.

1.1 **Oscillations without Dry Friction.** Begin with the value $\varphi_{\mathrm{m}} = 0$ corresponding to the absence of dry friction. Show that in this case the system displays the familiar behavior of a linear oscillator, i.e., simple harmonic oscillations with a constant amplitude in the absence of friction and with an exponentially decaying amplitude in the presence of viscous friction. The strength of viscous friction is characterized by the quality factor Q.

1.2 **Dry Friction after an Initial Displacement.** To display the role of dry friction clearly, choose a large value of the angle φ_{m} which determines the limits of the dead zone (say, 15 to 20 degrees), and let viscous friction be zero. Such conditions are somewhat unrealistic. They are far unlike the situation characteristic of measuring instruments using a needle, such as moving-coil galvanometers. These instruments are constructed so that the dead zone is as small as possible, and critical viscous damping is deliberately introduced in order to avoid taking a reading from an oscillating needle. When an instrument is critically damped, its moving system just fails to oscillate, and it comes to rest in the shortest possible time. If the dead zone is narrow, the needle stops at a position very close to the dial point which gives the true value of the measured quantity. Here, on the other hand, conditions are chosen to clarify the role of dry friction.

(a) What can you say about the succession of maximal deflections if damping is caused only by dry friction with the ideal z-characteristic? What is the law of their diminishing? How is the difference of consecutive maximal deflections related to the half-width of the dead zone?

(b) Let the angle φ_{m} that defines the boundaries of the stagnation zone be, say, 15°, the initial angle of deflection φ_0 be 160°, and the initial angular velocity be zero. Calculate the point of the dial at which the needle eventually comes to rest. How many semi-ellipses form the phase trajectory of this motion, from its initial point to the point at which the motion stops? Verify your predictions by simulating the motion on the computer.

(c) In the graph of the time dependence of the deflection angle, where are the midpoints of the half-cycles of the sinusoidal oscillations located? Note how these individual segments of the sine curves are joined to form a continuous plot of damped oscillations.

(d) In the graph of the angular velocity versus time, note the abrupt bends in the curve at the instants at which the midpoints abruptly replace one another. What is the reason for these bends? Prove that these instants are separated by half the period of harmonic oscillations in the absence of dry friction. (Note that points on the time scale of the graphs correspond to integral multiples of the period.)

1.3* **Dry Friction after an Initial Push.** Choose different initial conditions: let the initial deflection be zero, and the initial angular velocity be, say, $2\omega_0$ (where ω_0 is the natural frequency of oscillations). Use the same value $\varphi_{\rm m} = 15^\circ$ as above.

(a) Calculate the maximal deflection of the needle.

(b) To what position on the dial does the needle point when oscillations cease? How many turns are present in the complete phase trajectory of this motion? Verify your answer using a simulation experiment on the computer.

1.4* **Damping by Dry Friction at Various Initial Conditions.** Assuming the same width of the dead zone as above, calculate the maximal angle of deflection and the final position on the dial to which the needle points when oscillations cease, for the more complicated initial conditions:

(a) The initial deflection angle $\varphi(0) = 135^\circ$, and the initial angular velocity $\dot\varphi(0) = 1.5\omega_0$ (ω_0 is the natural frequency of the oscillator).

(b) The initial deflection angle $\varphi(0) = -135^\circ$, and the initial angular velocity $\dot\varphi(0) = 1.5\omega_0$.

Verify your calculated values in a simulation experiment on the computer.

1.5* **Energy Dissipation at Dry Friction.**

(a) The graph of the total mechanical energy versus the angle of deflection consists of rectilinear segments joining the slopes of the parabolic potential well (when you work in the section "Energy transformations" of the computer program). Suggest an explanation.

(b) Letting the initial angular velocity $\dot\varphi(0) = 2\omega_0$, where ω_0 is the natural frequency, and using energy considerations, calculate the entire angular path of the flywheel, excited from the midpoint of the dead zone by an initial push if the half-width of the dead zone $\varphi_{\rm m} = 10^\circ$.

1.6 **Oscillations in the Case of a Narrow Dead Zone.** Choose a small value for the angle $\varphi_{\rm m}$ (less than 5°), and set the initial angular displacement to be many times the width of the dead zone, $2\varphi_{\rm m}$.

(a) How many cycles does the flywheel execute before stopping?

(b) When the number of cycles is large, the plots clearly demonstrate the linear decay of the amplitude and the equidistant character of the loops in the phase diagram. What can you say about the time dependence of the total energy, averaged over a cycle?

2 Influence of Viscous Friction

2.1* **Transition of the Main Role from Viscous to Dry Friction.** When damping is caused both by dry and viscous friction, it is interesting to observe the change in the character of damping when the main contribution passes from viscous to dry friction.

Let the angle φ_{m} that determines the width of the dead zone be about 1° and let the quality factor Q which characterizes the strength of viscous friction be about 30. Let the initial angular deflection be 120° and the initial angular velocity be zero.

(a) Does dry or does viscous friction determine the initial damping effects?

(b) At what value of the amplitude does the character of damping change? How does this change manifest itself on the plots of time dependence of the angle of deflection and of the angular velocity? On the phase trajectory?

2.2* **Both Viscous and Dry Friction.** Let the boundaries of the stagnation interval be at $\varphi_{\mathrm{m}} = 10^\circ$ and the quality factor $Q = 5$. Let the initial velocity be $2\omega_0$ and the initial deflection be zero.

(a) Calculate the maximal angular deflection of the needle at these initial conditions. Verify your answer experimentally.

(b) What kind of friction, dry or viscous, initially dominates the damping of oscillations?

(c)** Let the boundaries of the stagnation zone be determined by the angle $\varphi_{\mathrm{m}} = 10^\circ$. Let the quality factor Q be 3, the initial deflection be 65°, and the initial angular velocity be$-2\omega_0$. Calculate the maximal angular deflection of the needle in the direction opposite the initial deflection. Verify your answer experimentally.

2.3 **Dry Friction and Critical Viscous Damping.**

(a) Choose the quality factor Q to be near the critical value 0.5 and investigate the character of damping experimentally. Where within the limits of the dead zone is the needle most likely to stop if the quality Q is slightly greater than the critical value? Give some physical explanation of your observations.

(b) Where would the needle stop if the quality factor Q is less than 0.5 (that is, if the system is overdamped)? Does the answer depend on the initial conditions?