# Free Oscillations of Linear Oscillator – Problems

## Summary of the Principal Formulas

The differential equation of a free linear torsion oscillator:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = 0.$$

The frequency and the period of free oscillations without friction (at $\gamma \ll \omega_0$):

$$\omega_0 = \sqrt{\frac{D}{J}}, \qquad T_0 = \frac{2\pi}{\omega_0}.$$

An oscillatory solution (valid at $\gamma < \omega_0$):

$$\varphi(t) = A_0 e^{-\gamma t}\cos(\omega_1 t + \delta_0),$$

where the constants $A_0$ and $\delta_0$ are determined by the initial conditions $\varphi(0)$, $\dot{\varphi}(0)$. The frequency $\omega_1$ of damped oscillations

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}.$$

An equivalent form of the general solution:

$$\varphi(t) = e^{-\gamma t}(C\cos\omega_1 t + S\sin\omega_1 t),$$

where the constants $C$ and $S$ are determined by the initial conditions. They are related to $A_0$ and $\delta_0$:

$$A_0 = \sqrt{C^2 + S^2}, \qquad \tan\delta_0 = -S/C.$$

In the case of weak damping ($\gamma \ll \omega_0$)

$$\omega_1 \approx \omega_0 - \gamma^2/(2\omega_0).$$

The decay time (during which the amplitude is reduced by the factor $e \approx 2.72$):

$$\tau = 1/\gamma.$$

A non-oscillatory motion at $\gamma = \omega_0$:

$$\varphi(t) = (C_1 t + C_2)e^{-\gamma t}.$$

The quality factor $Q$ of an oscillator:

$$Q = \pi \frac{\tau}{T_0} = \frac{\omega_0}{2\gamma}.$$

The number of oscillations, during which the amplitude is halved:

$$N_{1/2} = \frac{\ln 2}{\pi} Q = 0.22\, Q = \frac{Q}{4.53}.$$

The total mechanical energy of the oscillator consists of elastic potential energy of the strained spring and kinetic energy of the flywheel:

$$E = E_{\text{pot}} + E_{\text{kin}} = \frac{1}{2} D\varphi^2 + \frac{1}{2} J\dot{\varphi}^2.$$

The values of the potential energy and kinetic energy of the oscillator, averaged over a cycle, equal one another, each of them constituting one half the total energy:

$$\langle E_{\text{pot}} \rangle = \langle E_{\text{kin}} \rangle = \frac{1}{2} E = \frac{1}{4} D A_0^2 = \frac{1}{4} J \omega_0^2 A_0^2.$$

# 1 Free Undamped Oscillations

1.1 **The Initial Conditions and the Shape of the Plots.**

In the absence of friction a linear oscillator executes simple harmonic motion, which is characterized by purely sinusoidal time dependence of the angular displacement and of the angular velocity.

(a) What initial conditions give rise to oscillations of cosine time dependence, of sine time dependence? Suppose that you want to get oscillations with the angular amplitude of $90^\circ$. What initial angular displacement $\varphi(0) = \varphi_0$ at zero initial angular velocity $\dot{\varphi}(0) = 0$ ensures the desired amplitude?

(b) What initial angular velocity $\dot{\varphi}(0) = \Omega$ ought you to impart to the oscillator, at rest in the equilibrium position, in order to obtain the same amplitude of $90^\circ$? Remember, that the initial angular velocity $\Omega$ must be expressed for input in units of the natural frequency $\omega_0$. Verify your answer with a computer experiment, using the appropriate initial conditions.

1.2 **Maximal Deflection and Conservation of Energy.** Imagine exciting an oscillator initially at rest in the equilibrium position by a push which produces an initial angular velocity $\Omega = 2\omega_0$.

(a) Calculate the angle $\varphi_{\max}$ of maximal deflection using the law of the conservation of energy.

(b) Verify your result experimentally. Note that the simulation program performs the numerical integration of the differential equation independently of conservation laws, such as the conservation of energy. That is, these laws are not used in the program.

1.3 **The Phase Trajectory and the Initial Conditions.** Compare the motion of the representative point along the phase trajectory of a conservative oscillator with the time-dependent plots of the angle of deflection and of the angular velocity.

(a) How is the phase trajectory changed if you change the initial conditions?

(b) Does the direction of the motion of the representative point along the phase trajectory depend on the initial conditions?

(c) Is it possible that phase trajectories for different initial conditions coincide? If so, formulate the requirements for the coincidence.

1.4 **Elliptical and Circular Shape of the Phase Trajectory.**

(a) Prove analytically that the phase trajectory of a conservative linear oscillator is an ellipse with its center at the origin of the phase plane. What are the semiaxes of the ellipse?

(b) Show that the elliptical shape of the phase diagram of a conservative linear oscillator follows immediately from the law of the conservation of the energy.

(c) What scale on the axis of the ordinate (the angular velocity axis) of the phase plane produces a circular phase trajectory?

(d) Does the time interval during which the representative point passes along one loop of the phase trajectory depend on the initial conditions?

1.5 **The Phase Diagram and Energy Transformations.** Compare the phase trajectory with the plot of potential energy versus the angle of deflection. The positioning of plots on the display screen (if you open the window “Phase diagram”) is convenient for such comparison. Pay special attention to the positions of the extreme points on the phase trajectory and in the parabolic potential well. For the initial conditions $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \Omega$, what are the values of the potential energy and the kinetic energy at the extreme points and at the equilibrium position?

What are the extreme deflection $\varphi_{\max}$ and the maximal angular velocity $\omega_{\max}$ of the flywheel?

1.6 **The Shape and the Frequency of Energy Oscillations.** Consider the plots of the time dependence of kinetic energy and potential energy.

(a) What can you say about their maximal and average values? Compare these plots with the plots of the angular displacement and the angular velocity.

(b) At what frequency do the oscillations of each kind of energy occur? What are the limits (the extreme values) and the mean (averaged over a period) values of each kind of energy in these oscillations?

1.7 **The Phase Trajectories with the Same Energy.** Consider the oscillations of a conservative oscillator at different initial conditions but with the same total energy. What differences do you observe in the plots and the phase trajectories in these cases?

## 1.1 Damped Free Oscillations

2.1 **The Sequence of Maximal Deflections.** Under the action of a weak force of viscous friction, the sequence of maximal deflections of a free, damped linear oscillator forms a decreasing geometric progression: each consecutive maximal deflection is smaller than the preceding one by the same factor, $\exp(-\gamma T_0) \approx 1 - \gamma T_0$.

(a) Calculate the value of the quality factor $Q$ at which the amplitude halves during every two complete oscillations.

(b) Input this value in a computer experiment and verify the theoretically predicted constant ratio of successive maximal deflections. Note that this ratio does not depend on the initial conditions.

(c) Evaluate the increment of the period of oscillations at this value of the quality factor with respect to the period $T_0$ in the absence of friction (in percent). Can you detect the increment in the simulation experiment? The marks on the time axis correspond to integer numbers of periods $T_0 = 2\pi/\omega_0$ without friction.

2.2* **Maximal Deflection after an Initial Push.** Imagine, that we excite oscillations with an initial push which imparts an initial angular velocity of $2\omega_0$ to the flywheel in its equilibrium position.

(a) Calculate the first maximal deflection of the flywheel for the quality factor $Q = 5$.

(b) What will be the value of the subsequent extreme deflection which occurs in the direction opposite to the first? Verify your answers.

2.3** **Complex Initial Conditions.**

(a) Let the initial deflection of the torsion pendulum be $155$ degrees, and the initial angular velocity be $2\omega_0$. The quality factor $Q = 5$. Calculate the maximal deflection of the flywheel.

(b) With the same initial deflection ($155$ degrees) and the same quality factor $Q = 5$ as in the preceding item (a), calculate the maximal deflection of the flywheel, if the initial angular velocity equals $-2\omega_0$.

(c) Let the initial deflection of the torsion pendulum be $-155$ degrees. What initial angular velocity would ensure the maximal deflection of 155 degrees (to the opposite side), if the quality factor $Q = 20$?

2.4* **The Phase Trajectory of Damped Oscillations.** The phase trajectory of damped free oscillations for $Q > 0.5$ is a spiral which makes an infinite number of gradually shrinking loops around the focus located at the origin of the phase plane. This focus corresponds to the state of rest in the equilibrium position, and the phase trajectory approaches it asymptotically.

(a) How does the radius of these loops change while the curve approaches the focus?

(b) Does the time interval during which the representative point makes one revolution of the spiral change as the loops of the curve shrink?

2.5* **The Dissipation of Energy.** Compare the transformation of potential energy into kinetic energy (and vice versa) for free undamped oscillations in the absence of friction with that for free damped oscillations in the presence of viscous friction.

(a) Show, using a simulation experiment, that if $Q$ = 18.1, the amplitude is halved during four complete oscillations and the total energy is halved during two complete oscillations.

(b) Why is the dissipation of mechanical energy nonuniform during one cycle of oscillations? At what instants during a cycle is the time-rate of energy dissipation greatest and at what instants is it smallest?

## 1.2 Non-oscillatory Motion of the Pendulum

When viscous friction is strong ($Q \leq 0.5$), a disturbed system returns to the equilibrium position without oscillating. In the computer simulation, the needle asymptotically approaches the zero point from one side.

3.1* **Non-oscillatory Motion at Critical Damping.** Consider the case of critical damping, $\gamma = \omega_0$.

(a) Why is critical damping preferable in measuring instruments using a needle as an indicator? How might your answer apply to the suspension system in an automobile?

Show that the value of $Q$ in the case of critical damping is $0.5$.

(b) Calculate the maximal angle of deflection if the system, with $Q = 0.5$, receives an initial velocity $\Omega = 5\omega_0$ in the equilibrium position.

(c) In what lapse of time does the needle move towards this extreme point?

Verify your answers by simulating the experiment on the computer. Note that the needle approaches the equilibrium position from one side—it does not cross the zero point of the dial.

3.2 **Critical Damping.**

(a) Prove that the value $Q = 0.5$ ($\gamma = \omega_0$) is really critical. Do so by showing that at slightly greater values of $Q$, the needle of a perturbed oscillator executes heavily damped oscillations, slowly moving to and fro across the zero point of the dial. (Sound ticks at crossing the zero point may help you).

(b) For a critically damped system, express the constants $C_1$ and $C_2$ in the general solution $\varphi(t) = (C_1 t + C_2)\exp(-\gamma t)$ of the differential equation in terms of the initial displacement $\varphi(0) = \varphi_0$ and the initial angular velocity $\dot{\varphi}(0) = \Omega_0$.

(c) Is it possible for a critically damped system to move after an initial disturbance according to pure exponential law? If so, what initial conditions give rise to such motion? What is the phase trajectory of this motion? Prove your answers experimentally.

(d) At what initial conditions the flywheel of a disturbed critically damped system will cross the equilibrium position? For a given initial displacement $\varphi_0$, what initial angular velocity $\Omega$ should you impart to the flywheel of the critically damped oscillator in order it crossed the equilibrium position after a lapse of time $t = 3T_0$, where $T_0 = 2\pi/\omega_0$ is the natural period (the period of oscillations in the absence of friction)?

3.3* **Motion of an Overdamped System.**

(a) For arbitrary initial conditions ($\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \Omega_0$), express the values of arbitrary constants in the general solution of the differential equation for an overdamped system in terms of $\varphi_0$ and $\Omega_0$.

(b) At what initial conditions the motion of an overdamped system will be described by a monoexponential function of time? What are the phase trajectories that correspond to such motions?

(c) Explain, why at arbitrary initial conditions non-oscillatory motion of the flywheel towards the equilibrium position occurs more slowly and requires more time than at critical damping. Is it possible for an overdamped system to return to the equilibrium position faster than for the critically damped system with the same $\omega_0$? If so, what conditions of excitation ensure the motion?

(d) What is the principal difference between the phase trajectories corresponding to a non-oscillatory motion and those corresponding to damped oscillations?

(e) Is it possible for an overdamped system ($\gamma > \omega_0$) to cross the equilibrium position after excitation? If so, what initial conditions give rise to such motion? Is it possible for the oscillator to cross the equilibrium position more than once?