Torsion Spring Oscillator with Dry Friction

Manual

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Annotation. The manual includes a description of the simulated physical system and a summary of the relevant theoretical material for students as a prerequisite for the virtual lab "Torsion Spring Oscillator with Dry Friction." The manual includes also a set of theoretical and experimental problems to be solved by students on their own, as well as various assignments which the instructor can offer students for possible individual mini-research projects.

Contents

1 Summary of the Theory

1.1 General Concepts

This lab is aimed at investigation of free oscillations of a torsion spring pendulum damped by dry (Coulomb) friction. An idealized mathematical model of dry friction described by the so-called z-characteristic is assumed. In this model, the force of kinetic friction does not depend on speed and equals the limiting force of static friction. The physical system modelled here allows us to understand the origin of accidental errors in reading some measuring instruments.

1.2 The Physical System

The rotating component of the torsion spring oscillator is a balanced flywheel (a rigid rod with two equal weights) whose center of mass lies on the axis of rotation. A spiral spring, one end fixed and the other end attached to the flywheel, flexes when the flywheel is turned. The spring provides a restoring torque whose magnitude is proportional to the angular displacement of the flywheel from the equilibrium position. Figure 1 shows the image of the simulated system as it appears on the computer screen.

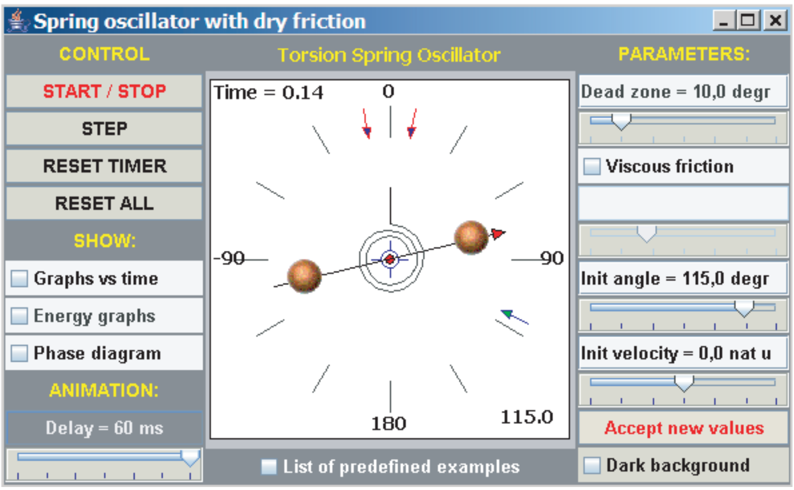

Figure 1: The image of the torsion spring oscillator as it appears on the computer screen.

The dynamical behavior of such a system under the influence of viscous friction (for which the torque is proportional to the angular velocity) is discussed in the manual for the lab "Free oscillations of a linear torsion pendulum." The reader should be familiar with this material before proceeding with this lab.

When friction is *viscous*, free oscillations of a spring pendulum are described by a *linear* differential equation. The amplitude of such oscillations decreases exponentially with time. That is, the consecutive maximal deflections of the oscillator from its equilibrium position are in a diminishing geometric progression because their ratio is constant.

In principle such oscillations continue indefinitely, their amplitude asymptotically approaching zero. However, it is convenient to characterize the duration of exponential damping by a *decay time* τ. This conventional time of damping τ is the lapse of time during which the amplitude of free oscillations decreases by a factor of $e \approx 2.72$.

The exponential character of damping caused by viscous friction follows from the proportionality of friction to velocity. Some other relationship between friction and velocity produces damping with different characteristics.

The case of *dry* or *Coulomb* friction has important practical applications. In this case, as long as the system is moving, the magnitude of dry friction is very nearly constant and its direction is opposite that of the velocity. An idealized simplified characteristic of dry friction (called the z-characteristic) is shown in figure 2. The graph shows dependence of the frictional torque on the velocity of rotation. Here the magnitude of friction is constant, but its direction changes each time the direction of the velocity changes. When the system is at rest, the torque of static dry friction takes on any value from $-N_{\max}$ to $N_{\max}$. The actual value depends on the friction needed to balance the other forces exerted on the system. The magnitude of the torque of kinetic dry friction is assumed in this model to be equal to the limiting torque $N_{\max}$ of static friction.

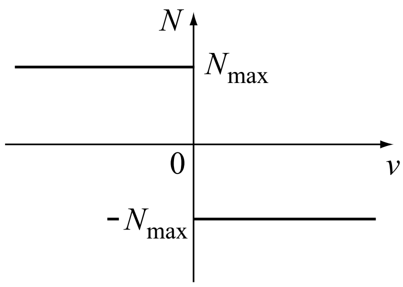

Figure 2: An idealized characteristic of dry friction (z-characteristic).

In real physical systems dry friction is characterized by a more complicated dependence on velocity. The limiting force of static friction is usually greater than the force of kinetic friction. When the speed of a system increases from zero, kinetic friction at first decreases, reaches a minimum at some speed, and then gradually increases with a further increase in speed. These peculiarities are ignored in the idealized z-characteristic of dry friction. Nevertheless, this idealization helps us to understand many essential properties of oscillatory processes in real physical systems.

Because the magnitude of the torque of static friction can assume any value up to $N_{\max}$, there is a range of values of displacement called the *stagnation interval* or *dead zone* in which static friction can balance the restoring elastic force of the strained spring. At any point within this interval the system can be at rest in a state of neutral equilibrium, in contrast to a single position of stable equilibrium provided by the spring in the case of viscous friction. If the velocity becomes zero at some point of the dead zone, the system remains at rest there. The boundaries of the dead zone are indicated by small arrows on the dial in figure 1.

The stagnation interval extends equally to either side of the point at which the spring is unstrained. The stronger the dry friction in the system, the more extended the stagnation interval. The boundaries of the interval are determined by the limiting torque N_{max} of static friction.

An important feature of oscillations damped by dry friction is that motion ceases after a *finite* number of cycles. As the system oscillates, the sign of its velocity changes periodically, and each subsequent change occurs at a smaller displacement from the mid-point of the stagnation interval. Eventually the turning point of the motion occurs within the stagnation interval, in which static friction can balance the restoring force of the spring, and so the motion abruptly stops. The exact position in the stagnation interval at which this event occurs, depends on the initial conditions, which may vary from one situation to the next.

These characteristics are typical of various mechanical systems with dry friction. For example, dry friction may be encountered in measuring instruments, such as a moving-coil galvanometer, in which readings are taken with a needle. In the galvanometer, a light coil of wire is pivoted between the poles of a magnet. When a current flows through the coil, it turns against a spiral return spring. If the coil axis is fixed in unlubricated bearings and hence experiences dry friction, the needle of the coil may come to rest and show to any point of the stagnation interval on either side of the dial point which gives the true value of the measured quantity. So we can now understand one of the reasons that random errors inevitably occur in the readings of moving-coil measuring instruments. The larger the dry friction, the larger the errors of measurement.

1.3 The Differential Equation of the Oscillator

The rotating flywheel of the torsion oscillator is simultaneously subjected to the action of the restoring torque $-D\varphi$ produced by the spring, and of the torque N_{fr} of kinetic dry friction. The differential equation describing the motion of the flywheel, whose moment of inertia is J, is thus

$$J\ddot{\varphi} = -D\varphi + N_{\mathrm{fr}}. \tag{1}$$

According to the idealized z-characteristic of dry friction, the torque N_{fr} is directed oppositely to the angular velocity $\dot{\varphi}$, and is constant in magnitude while the flywheel is moving, but may have any value in the interval from $-N_{\mathrm{max}}$ up to N_{max} while the flywheel is at rest:

$$N_{\mathrm{fr}}(\dot{\varphi}) = \begin{cases} -N_{\mathrm{max}} & \text{for} \quad \dot{\varphi} > 0, \\ N_{\mathrm{max}} & \text{for} \quad \dot{\varphi} < 0, \end{cases} \tag{2}$$

or $N_{\mathrm{fr}} = -N_{\mathrm{max}}\,\mathrm{sign}(\dot{\varphi})$. Here N_{max} is the limiting value of the static frictional torque. It is convenient to express the value N_{max} in terms of the maximal possible deflection angle φ_{m} of the flywheel at rest:

$$N_{\mathrm{max}} = D\varphi_{\mathrm{m}}. \tag{3}$$

The angle φ_{m} corresponds to the boundary of the stagnation interval.

The differential equation, Eq. (1), in the general case of an oscillator with dry friction, is *nonlinear* because the torque $N_{\rm fr}(\dot\varphi)$ abruptly changes when the sign of $\dot\varphi$ changes at the extreme points of oscillation, and because when the flywheel moves, the torque is usually not constant. But in the idealized case of the z-characteristic we may consider the following two *linear* equations instead of Eq. (1):

$$J\ddot\varphi = -D(\varphi + \varphi_{\rm m}) \qquad \text{for} \quad \dot\varphi > 0, \tag{4}$$

$$J\ddot\varphi = -D(\varphi - \varphi_{\rm m}) \qquad \text{for} \quad \dot\varphi < 0. \tag{5}$$

Whenever the sign of the angular velocity $\dot\varphi$ changes, the pertinent equation of motion also changes. The nonlinear character of the problem reveals itself in alternate transitions from one of the linear equations, Eqs. (4)–(5), to the other.

The solution to Eqs. (4)–(5) which corresponds to a set of given initial conditions can be found by using the method of the stage-by-stage integration of each of the linear equations for the half-cycle during which the direction of motion is unchanged. These solutions are then joined together at the instants of transition from one equation to the other in such a way that the displacement at the end point of one half-cycle becomes the initial displacement at the beginning of the next half-cycle. This array of solutions continues until the end point of a half-cycle lies within the dead zone.

If in addition to dry friction the oscillator also experiences viscous friction, we must add to the equations of motion, Eqs. (4)–(5), one more term proportional to the angular velocity $\dot\varphi$:

$$\ddot\varphi = -\omega_0^2(\varphi + \varphi_{\rm m}) - 2\gamma\dot\varphi \qquad \text{for} \quad \dot\varphi > 0, \tag{6}$$

$$\ddot\varphi = -\omega_0^2(\varphi - \varphi_{\rm m}) - 2\gamma\dot\varphi \qquad \text{for} \quad \dot\varphi < 0. \tag{7}$$

Here $\omega_0^2 = D/J$ is the squared natural frequency of the oscillator (the frequency of oscillations in the absence of friction), and γ is the damping constant. It is convenient to characterize viscous friction by the dimensionless quality factor, $Q = \omega_0/2\gamma$.

1.4 Damping Caused by Dry Friction

In order to discover the fundamental characteristics of oscillations which are damped under the action of dry friction, we shall first assume that viscous friction is absent ($\gamma = 0$).

At the initial instant $t = 0$, let the flywheel be displaced to the right (clockwise) from the equilibrium position so that $\varphi(0) > 0$. If this displacement exceeds the boundary of the stagnation interval, i.e., if $\varphi(0) > \varphi_{\rm m}$, the flywheel, being released without a push, begins moving to the left ($\dot\varphi < 0$), and its motion is described by Eq. (5). The solution to Eq. (5) with the given initial conditions ($\varphi(0) = \varphi_0$, $\dot\varphi(0) = 0$) is simple harmonic motion whose frequency is ω_0. The midpoint of the motion is $\varphi_{\rm m}$. This point coincides with the right-hand boundary of the stagnation interval. The displacement $\varphi_{\rm m}$ of the midpoint from zero is caused by the constant torque of kinetic friction. This torque is directed to the right (clockwise) while the flywheel is moving to the left. The amplitude of this oscillation about the

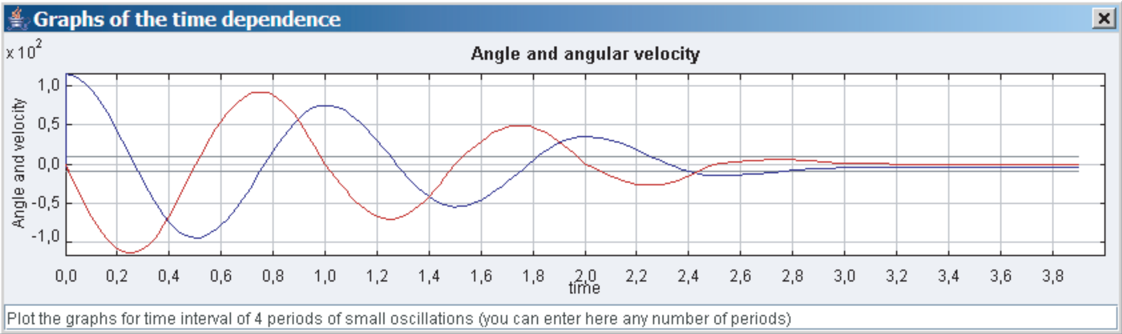

Figure 3: Damping of oscillations caused by dry friction

midpoint φ_{m} is $\varphi_0 - \varphi_{\mathrm{m}}$. The first segment of the graph in figure 3 (the first half-cycle of the sine curve, whose midpoint is at a height of φ_{m} above the abscissa axis) is a plot of this part of the motion.

Since the amplitude of the first half-cycle is $\varphi_0 - \varphi_{\mathrm{m}}$, the extreme left position of the flywheel at the end of the half-cycle is $\varphi(0) - 2\varphi_{\mathrm{m}}$. When the flywheel reaches this position, its velocity is momentarily zero, and it starts to move to the right. Since its angular velocity $\dot{\varphi}$ is subsequently positive, we must now consider Eq. (4). The values of φ and $\dot{\varphi}$ at the end of the preceding half-cycle are taken as the initial conditions for this half-cycle. Thus the subsequent motion is again a half-cycle of harmonic oscillation with the same frequency ω_0 as before but with the midpoint $-\varphi_{\mathrm{m}}$ displaced to the left, i.e., with the midpoint at the left-hand boundary of the stagnation interval. This displacement is caused by the constant torque of kinetic friction, whose direction was reversed when the direction of motion was reversed. The amplitude of the corresponding segment of the sine curve is $\varphi_0 - 3\varphi_{\mathrm{m}}$.

Continuing this analysis half-cycle by half-cycle, we see that the flywheel executes harmonic oscillations about the midpoints alternately located at φ_{m} and $-\varphi_{\mathrm{m}}$. The frequency of each cycle is the natural frequency ω_0, and so the duration of each full cycle equals the period $T_0 = 2\pi/\omega_0$ of free oscillations in the absence of friction.

The joining together of these sinusoidal segments, whose midpoints alternate between the boundaries of the stagnation interval, produces the curve that describes oscillatory motion damped by dry friction (figure 3). The maximal deflection decreases after each full-cycle of these oscillations by a constant value equal to the doubled width of the stagnation interval (i.e., by the value $4\varphi_{\mathrm{m}}$). The oscillation continues until the end point of some next in turn segment of the sine curve occurs within the dead zone $(-\varphi_{\mathrm{m}},\ \varphi_{\mathrm{m}})$.

Thus, in the case of dry friction, consecutive maximal deflections diminish linearly in a decreasing arithmetic progression, and the motion stops after a final number of cycles, in contrast to the case of viscous friction, for which the maximal displacements decrease exponentially in a geometric progression, and for which the motion continues indefinitely.

1.5 The Phase Trajectory

The character of oscillations in the presence of dry friction is given clearly by the phase trajectory shown in figure 4. The system is initially at rest ($\dot{\varphi}(0) = 0$) and displaced to the right ($\varphi(0) = \varphi_0 > \varphi_{\rm m}$). This initial state is represented by the point on the curve which lies to the extreme right on the horizontal axis, the φ-axis. The portion of the phase trajectory lying below the horizontal axis represents the motion during the first half-cycle, when the flywheel is moving to the left. This curve is the lower half of an ellipse (or of a circle if the scales have been chosen appropriately) whose center is at the point $\varphi_{\rm m}$ on the horizontal axis. This point corresponds to the right-hand boundary of the stagnation interval.

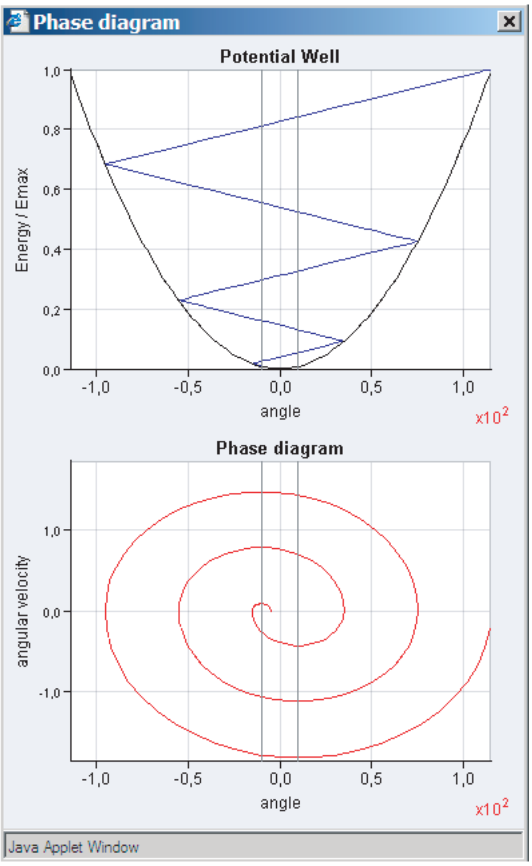

Figure 4: The phase diagram of damping caused by dry friction

The second half-cycle, when the flywheel is moving to the right, is represented by half an ellipse lying above the φ-axis, where angular velocities are positive. The center of this second semi-ellipse is at the point $-\varphi_{\rm m}$, on the φ-axis. The complete phase trajectory is formed by such increasingly smaller semi-ellipses, alternately centered at $\varphi_{\rm m}$ and $-\varphi_{\rm m}$. The diameters of these consecutive semi-ellipses lie along the φ-axis and decrease each half-cycle by $2\varphi_{\rm m}$. The phase trajectory terminates on the φ-axis at the point at which the curve meets the φ-axis inside the dead zone (the portion of the φ-axis lying between $\varphi_{\rm m}$ and $-\varphi_{\rm m}$).

This phase trajectory is to be compared with that of the oscillator acted upon by viscous friction. In the latter case, the curve spirals around a focal point located at the origin of the phase plane. The curve consists of an infinite number of turns which gradually become smaller and which approach the focus asymptotically. In the present case of dry friction, the loops of the phase curve are equidistant. The phase trajectory consists of a finite number of cycles and terminates at the point at which it meets the segment of the φ-axis between the points $-\varphi_{\rm m}$ and $\varphi_{\rm m}$.

If dry friction in the system is accompanied by a rather weak viscous friction ($\gamma < \omega_0$), the semi-ellipses become distorted and their axes shrink during the motion. The loops of the phase trajectory are no longer equidistant. Nevertheless their shrinking does not last indefinitely: the phase trajectory in this case also terminates after some finite number of turns around the origin when it reaches the stagnation interval on the φ-axis.

1.6 Energy Transformations

While the flywheel is rotating in one direction, the torque $N_{\rm max}$ of kinetic friction, independent of the velocity, is constant, and the total energy of the oscillator decreases linearly with the angular displacement, φ, of the flywheel. This linear dependence of the total energy on φ is clearly indicated in the upper plot in figure 4, where the parabolic potential well of the elastic spring is shown. The representing point whose ordinate gives the total energy $E(\varphi)$ and whose abscissa gives the angular displacement of the flywheel, travels in the course of time between the slopes of this well, gradually descending to the bottom of the well. The trajectory of this point consists of rectilinear segments lying between the sides of the well. These segments are straight because the negative work done by the force of dry friction is proportional to the angle of rotation, $\Delta\varphi$. The amount of this work $|N_{\rm max}\Delta\varphi|$ equals the decrease $-\Delta E$ of total energy.

However, the dependence of total energy on time, $E(t)$, is not linear because the rotation of the flywheel is nonuniform. The time rate of dissipation of the total energy, $-dE/dt$, is proportional to the magnitude of the angular velocity, $|\dot{\varphi}(t)|$. Thus, the greatest rate of dissipation of mechanical energy through friction occurs when the magnitude of the angular velocity, $\dot{\varphi}$, is greatest, that is, when the flywheel crosses the boundaries of the dead zone. Near the points of extreme deflection, where the angular velocity is near zero, the time rate of dissipation of mechanical energy is smallest. Typical plots of energy transformation are shown in figure 5.

Unlike the case of viscous friction, the oscillator with dry friction may retain some mechanical energy E_f at the termination of the motion. Such occurs if the final angular displacement (within the dead zone) is not at the midpoint of the stagnation interval. Then the spring remains strained, and its elastic potential energy is not zero. The remaining energy does not exceed the value $D\varphi_{\rm m}^2/2 = N_{\rm max}\varphi_{\rm m}/2$.

When the initial excitation is large enough, that is, when the initial energy is much greater than $D\varphi_{\rm m}^2/2$, the oscillator executes a large number of cycles before the oscillations cease. In this case it is reasonable to consider the total energy averaged over the period of an oscillation, $\langle E(t)\rangle$. The decrease of $\langle E(t)\rangle$ during a large number of cycles depends quadratically on the lapse of time because the amplitude

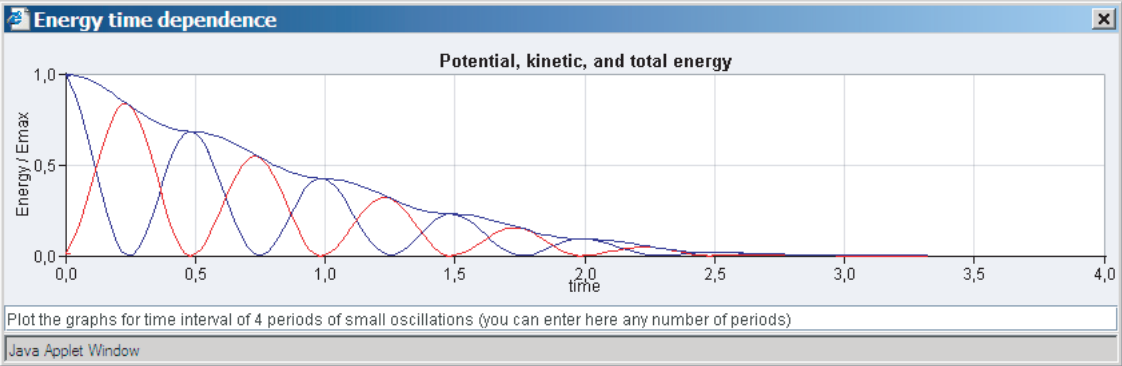

Figure 5: Energy transformations in oscillations damped by dry friction

of oscillation decreases linearly with time and because the averaged total energy is proportional to the square of the amplitude.

If we let t_f be the final moment when oscillations cease, then at the time t the averaged total energy $\langle E(t)\rangle$ is proportional to $(t - t_f)^2$. This statement (which clearly applies only for $t < t_f$) is exactly true only when the flywheel comes to rest at the center of the stagnation interval. However, even if such is not the case and there is a residual potential energy stored in the spring after the motion ceases, the statement is approximately true.

1.7 The Role of Viscous Friction

In real systems dry friction is always accompanied to some extent by viscous friction. The damping of oscillations in this case can also be investigated by the above-described method, namely by the stage-by-stage solving of Eqs. (6) and (7) and by using the final mechanical state (the angular displacement and velocity) of every half-cycle as the initial conditions for the next half-cycle. That is, the solutions are joined by equating their angular displacements and angular velocities (always zero) at the boundaries.

The clearest representation of the mechanical evolution of the system experiencing both dry and viscous friction is given by a phase diagram. Unlike the case of pure dry friction, the path in phase space is no longer a series of diminishing semi-ellipses (or semicircles) with alternating centers. Instead the phase trajectory consists of the shrinking alternating halves of spiral loops that are characteristic of a linear damped oscillator. The focal points of these spirals alternate between the boundaries of the stagnation interval.

To compare the relative importance of viscous versus dry friction, we consider below the decrease in amplitude caused by each of these effects during one complete cycle.

It was established above that under the action of dry friction this decrease equals the constant value of the doubled width of the stagnation interval $4\varphi_{\mathrm{m}}$. On the other hand, viscous friction decreases the amplitude of the oscillation during a complete cycle by an amount which is not constant but rather is proportional to the amplitude. Indeed, for $\gamma T_0 \ll 1$, i.e., for rather large values of the quality factor

Q, expression for the decrease Δa during one period T_0 in the momentary amplitude a due to viscous friction can be expanded in a series:

$$\Delta a = a(1 - e^{-\gamma T_0}) \approx a\gamma T_0 = a\gamma \frac{2\pi}{\omega_0} = \frac{\pi a}{Q}. \tag{8}$$

Equating Δa to the doubled width $4\varphi_{\mathrm{m}}$ of the stagnation interval, we find the amplitude $\tilde{a}$ which delimits the predominance of one type of friction over the other:

$$\tilde{a} = \frac{4\varphi_{\mathrm{m}}}{\gamma T_0} = \frac{4}{\pi}\varphi_{\mathrm{m}} Q \approx \varphi_{\mathrm{m}} Q. \tag{9}$$

If the actual amplitude is greater than $\tilde{a}$, the effect of viscous friction dominates. Conversely, if the actual amplitude is less than $\tilde{a}$, the effect of dry friction dominates.

When the initial excitation of the oscillator is great enough, the amplitude may exceed the value $\tilde{a} \approx Q\varphi_{\mathrm{m}}$. In this instance, the initial damping of the oscillations is influenced mainly by viscous friction. This case may be illustrated in the phase diagram. The decrement in the width of several initial loops of the phase trajectory (caused by viscous friction) is greater than the separation of the centers of adjoining half-loops (i.e., the decrement exceeds the width of the stagnation interval). It is clear that in this case the shrinking of the spiral caused by viscous friction is more influential in showing the effects of damping than is the alternation of the centers of half-loops caused by dry friction.

When the value of a falls below that of $\tilde{a}$ (when $a < \tilde{a} = Q\varphi_{\mathrm{m}}$), the effects of dry friction dominate. In the phase plane this dominance produces a trajectory of consecutive half-loops whose centers alternately jump between the ends of the stagnation interval, $-\varphi_{\mathrm{m}}$ and φ_{m}, until the phase trajectory reaches the segment of the φ-axis in the stagnation interval.

When viscous friction is strong, that is, when values of the quality factor Q are less than the critical value of 0.5 (when $\gamma > \omega_0$), and when the initial displacement of the flywheel $\varphi(0)$ lies beyond the boundaries of the stagnation interval, $|\varphi(0)| > \varphi_{\mathrm{m}}$, the needle of the released flywheel moves without oscillating toward the point of the dial which corresponds to the nearest boundary of the stagnation interval. At this point the flywheel stops turning.

2 Questions, Problems, Suggestions

The preceding analysis of the behavior of the oscillator under the influence of dry friction is based upon the method of the stage-by-stage analytic integration of the differential equations which describe the system. These equations are linear for the time intervals occurring between consecutive extreme deflections. These intervals are bounded by the instants at which the angular velocity is zero. The complete solution is obtained by joining together these half-cycle solutions for consecutive time intervals. On the other hand, the computer simulation of the torsion pendulum in this software package is based on the numerical integration of differential equations (the Runge—Kutta method to fourth order). To answer the questions below, you may apply the analysis described above. Then you can verify your analytic results by simulating the experiment on the computer.

2.1 Damping Caused by Dry Friction

The strength of dry friction in the system is characterized by the width of the dead zone. This interval is defined in the program when you input the value of the angle φ_m which sets the limits of the dead zone on both sides of the middle position at which the spring is unstrained. Total width of this dead zone is $2\varphi_\mathrm{m}$. The value of φ_m must be expressed in degrees.

1.1 **Oscillations without Dry Friction.** Begin with the value $\varphi_\mathrm{m} = 0$ corresponding to the absence of dry friction. Show that in this case the system displays the familiar behavior of a linear oscillator, i.e., simple harmonic oscillations with a constant amplitude in the absence of friction and with an exponentially decaying amplitude in the presence of viscous friction. The strength of viscous friction is characterized by the quality factor Q.

1.2 **Dry Friction after an Initial Displacement.** To display the role of dry friction clearly, choose a large value of the angle φ_m which determines the limits of the dead zone (say, 15 to 20 degrees), and let viscous friction be zero. Such conditions are somewhat unrealistic. They are far unlike the situation characteristic of measuring instruments using a needle, such as moving-coil galvanometers. These instruments are constructed so that the dead zone is as small as possible, and critical viscous damping is deliberately introduced in order to avoid taking a reading from an oscillating needle. When an instrument is critically damped, its moving system just fails to oscillate, and it comes to rest in the shortest possible time. If the dead zone is narrow, the needle stops at a position very close to the dial point which gives the true value of the measured quantity. Here, on the other hand, conditions are chosen to clarify the role of dry friction.

(a) What can you say about the succession of maximal deflections if damping is caused only by dry friction with the ideal z-characteristic? What is the law of their diminishing? How is the difference of consecutive maximal deflections related to the half-width of the dead zone?

(b) Let the angle φ_m that defines the boundaries of the stagnation zone be, say, 15°, the initial angle of deflection φ_0 be 160°, and the initial angular velocity be zero. Calculate the point of the dial at which the needle eventually comes to rest. How many semi-ellipses form the phase trajectory of this motion, from its initial point to the point at which the motion stops? Verify your predictions by simulating the motion on the computer.

(c) In the graph of the time dependence of the deflection angle, where are the midpoints of the half-cycles of the sinusoidal oscillations located? Note how these individual segments of the sine curves are joined to form a continuous plot of damped oscillations.

(d) In the graph of the angular velocity versus time, note the abrupt bends in the curve at the instants at which the midpoints abruptly replace one another. What is the reason for these bends? Prove that these instants are separated by half the period of harmonic oscillations in the absence of dry friction. (Note that points on the time scale of the graphs correspond to integral multiples of the period.)

1.3* **Dry Friction after an Initial Push.** Choose different initial conditions: let the initial deflection be zero, and the initial angular velocity be, say, $2\omega_0$ (where ω_0 is the natural frequency of oscillations). Use the same value $\varphi_\mathrm{m} = 15^\circ$ as above.

(a) Calculate the maximal deflection of the needle.

(b) To what position on the dial does the needle point when oscillations cease? How many turns are present in the complete phase trajectory of this motion? Verify your answer using a simulation experiment on the computer.

1.4* **Damping by Dry Friction at Various Initial Conditions.** Assuming the same width of the dead zone as above, calculate the maximal angle of deflection and the final position on the dial to which the needle points when oscillations cease, for the more complicated initial conditions:

(a) The initial deflection angle $\varphi(0) = 135°$, and the initial angular velocity $\dot{\varphi}(0) = 1.5\omega_0$ (ω_0 is the natural frequency of the oscillator).

(b) The initial deflection angle $\varphi(0) = -135°$, and the initial angular velocity $\dot{\varphi}(0) = 1.5\omega_0$.

Verify your calculated values in a simulation experiment on the computer.

1.5* **Energy Dissipation at Dry Friction.**

(a) The graph of the total mechanical energy versus the angle of deflection consists of rectilinear segments joining the slopes of the parabolic potential well (when you work in the section "Energy transformations" of the computer program). Suggest an explanation.

(b) Letting the initial angular velocity $\dot{\varphi}(0) = 2\omega_0$, where ω_0 is the natural frequency, and using energy considerations, calculate the entire angular path of the flywheel, excited from the midpoint of the dead zone by an initial push if the half-width of the dead zone $\varphi_{\mathrm{m}} = 10°$.

1.6 **Oscillations in the Case of a Narrow Dead Zone.** Choose a small value for the angle φ_{m} (less than $5°$), and set the initial angular displacement to be many times the width of the dead zone, $2\varphi_{\mathrm{m}}$.

(a) How many cycles does the flywheel execute before stopping?

(b) When the number of cycles is large, the plots clearly demonstrate the linear decay of the amplitude and the equidistant character of the loops in the phase diagram. What can you say about the time dependence of the total energy, averaged over a cycle?

2.2 Influence of Viscous Friction

2.1* **Transition of the Main Role from Viscous to Dry Friction.** When damping is caused both by dry and viscous friction, it is interesting to observe the change in the character of damping when the main contribution passes from viscous to dry friction.

Let the angle φ_{m} that determines the width of the dead zone be about $1°$ and let the quality factor Q which characterizes the strength of viscous friction be about 30. Let the initial angular deflection be $120°$ and the initial angular velocity be zero.

(a) Does dry or does viscous friction determine the initial damping effects?

(b) At what value of the amplitude does the character of damping change? How does this change manifest itself on the plots of time dependence of the angle of deflection and of the angular velocity? On the phase trajectory?

2.2* **Both Viscous and Dry Friction.** Let the boundaries of the stagnation interval be at $\varphi_{\mathrm{m}} = 10°$ and the quality factor $Q = 5$. Let the initial velocity be $2\omega_0$ and the initial deflection be zero.

(a) Calculate the maximal angular deflection of the needle at these initial conditions. Verify your answer experimentally.

(b) What kind of friction, dry or viscous, initially dominates the damping of oscillations?

(c)** Let the boundaries of the stagnation zone be determined by the angle $\varphi_{\mathrm{m}} = 10°$. Let the quality factor Q be 3, the initial deflection be $65°$, and the initial angular velocity be$-2\omega_0$. Calculate the maximal angular deflection of the needle in the direction opposite the initial deflection. Verify your answer experimentally.

2.3 **Dry Friction and Critical Viscous Damping.**

(a) Choose the quality factor Q to be near the critical value 0.5 and investigate the character of damping experimentally. Where within the limits of the dead zone is the needle most likely to stop if the quality Q is slightly greater than the critical value? Give some physical explanation of your observations.

(b) Where would the needle stop if the quality factor Q is less than 0.5 (that is, if the system is overdamped)? Does the answer depend on the initial conditions?

2.3 Supplement: Summary of the Principal Formulas

The differential equation of motion of an oscillator acted upon by dry friction:

$$J\ddot{\varphi} = -D(\varphi + \varphi_{\mathrm{m}}) \qquad \text{for} \qquad \dot{\varphi} > 0,$$
$$J\ddot{\varphi} = -D(\varphi - \varphi_{\mathrm{m}}) \qquad \text{for} \qquad \dot{\varphi} < 0,$$

where φ_m is the angle corresponding to the boundaries of the dead zone. If in addition, viscous friction is present, a term proportional to the angular velocity is also present:

$$\ddot{\varphi} = -\omega_0^2(\varphi + \varphi_{\mathrm{m}}) - 2\gamma\dot{\varphi} \qquad \text{for} \qquad \dot{\varphi} > 0,$$
$$\ddot{\varphi} = -\omega_0^2(\varphi - \varphi_{\mathrm{m}}) - 2\gamma\dot{\varphi} \qquad \text{for} \qquad \dot{\varphi} < 0,$$

where ω_0 is the natural frequency of oscillations in the absence of friction:

$$\omega_0^2 = \frac{D}{J}.$$

The damping factor γ that characterizes the viscous friction is related to the quality factor Q by the equation:

$$Q = \frac{\omega_0}{2\gamma}.$$

The boundary value of the amplitude that delimits the two cases in which the effects either of viscous friction or of dry friction predominate:

$$a = \frac{4\varphi_{\mathrm{m}}}{\gamma T} = \frac{4}{\pi}\varphi_{\mathrm{m}}Q \approx \varphi_{\mathrm{m}}Q.$$